Journal of Global Positioning Systems (2024)
Vol. 20, Joint No. 1 & No. 2: 69-87
DOI:10.5081/jgps.20.1.69

Journal of Global Positioning Systems
2024
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Precision Evaluation in Discrete Kalman Filtering: A Posteriori Perspective

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Abstract: This manuscript is focused on standardizing the process of the a posteriori precision evaluation in discrete Kalman filtering. Although the a posteriori precision evaluation of the solution was considered as indispensable within the method of least squares, the solution of a Kalman filter shows a lack of a posteriori precision evaluation for too long. Even worse, there often exists appalling confusion about what is considered as the a posteriori precision of the solution in Kalman filtering. The authors hereto propose to put the a posteriori precision evaluation of the solution into practice at four different levels in Discrete Kalman filtering through estimating: (i) the a posteriori variance of unit weight (or reference variance), (ii) the separate a posteriori variance factors for the process and measurement noise vectors, respectively, (iii) the individual a posteriori variance factors for the independent noise groups, and (iv) the individual a posteriori variance factors (or components) for the independent process noise factors and measurement types. A working example is presented to illustrate the proposed a posteriori precision evaluation in Kalman filtering using a road test based on the double-differenced GPS L1 C/A, L1 and L2 carrier phases and the specific force and angular rate measurements from an MEMS IMU. With the rapidly increasing utilization of the Kalman filter in modern applications, the inclusion of the proposed a posteriori solution precision evaluation, as part of the standard solution, in discrete Kalman filtering is not only necessary, but also can be expected to happen soon within our grasp.

KEY WORDS: a posteriori precision evaluation, Kalman filtering, variance factor (variance of unit weight), residuals, redundancy contribution, variance component estimation.

1. Introduction

"Data fusion describes different methods and techniques for combining data, information, and knowledge in order to improve data quality, reduce uncertainty, extract essential features and provide statistics and analytics, …" [ScienceDirect, 2025]. Undoubtedly, estimation methods and data (inclusive of information and knowledge) are the linchpins in data fusion. "Estimation, which fits in between the problems of measurement and validation, deals with the determination of those physical quantities that cannot be measured from those that can be measured" [Mendel, 1999]. Data are a series of observations, measurements, or facts [collinsdictionary.com]. A typical application is a multi-sensor integrated kinematic positioning and navigation system, the essential component of many applications such as autonomous car driving, unmanned aerial vehicles, and direct georeferencing technology for automatic geospatial data acquisition at large. Among different estimation techniques, the Kalman filter definitely belongs to the most significant for estimating the state of a dynamic system from noisy measurements toward achieving optimal solution. The general consensus is that measurements are decisive, no matter with which optimal estimation techniques, for example, least squares method, minimum variance, or others.

In Kalman filtering, one seeks for the best possible estimate of the state vector of a system through combining its system model (for prediction) with measurements on the ground of the principle of minimum variance. With both of the system and measurement models, the a priori stochastic model for them must be presumed before conducting the estimation, which describes the inherent randomness or uncertainty in both the being observed system and the being acquired measurements, specifically associated with the initial state vector and process noises of the

Editor-In-Charge: Dr. Ling Yang

system and the measurement noises. Such stochastic model is paramount of importance in Kalman filtering.

It is well known that the a posteriori precision and accuracy evaluation has been an inalienable part of least squares method [Wright and Hayford, 1906; Helmert, 1907; WCSM, 1959; Wells and Krakiwsky, 1971; Cross, 1983; Koch, 1987; Caspary 1988; Cui et al, 1993; Taylor, 1997; Rao and Toutenberg, 1999; Ghilani and Wolf, 2006; Kariya and Kurata, 2004; Wang et al, 2019]. But this topic barely appears as a standard part of the solution in Kalman filtering [Anderson, 1979; Brown and Huang, 2012; Chui and Chen, 2009; Eubank, 2009; Gelb, 1974; Grewal and Andrews, 2008; Salzmann, 1988; Simon, 2006; Teunissen et al, 2021; Zarchan, 2005; etc.]. There even often exists some confusion elementarily about what represents the a posteriori precision of the estimated state vector in Kalman filtering. The variance-covariance propagation for the optimally estimated state vector from time to time is commonly called the a posteriori precision (or accuracy), which is, however, dominantly dependent on the a priori stochastic model and has nothing to do with the actual behavior of system process and measurement noises associated with the being processed data. This deficiency in the solution formulation of a Kalman filter is incomprehensible either theoretically or practically. Fortunately, quite a few antecessors did touch on this problem to a certain extend [Pelzer, 1987; Koch, 1990; Tao, 1992; etc.]. The a posteriori variance of unit weight was introduced to geodetic deformation analysis wherever geodesists utilized the discrete Kalman filter for processing their multi-epoch repeated observations [Pelzer, 1987; Tao, 1992]. Yu et al (1988) proposed an approach to incorporate variance component estimation in the filtering process in monitoring networks. Furthermore, Koch [1990] clearly stated the problem where the covariance matrix $\boldsymbol{Q}(k)$ of the process noise vector and the covariance matrix $\boldsymbol{R}(k)$ of the measurement vector could have their unknown variance factors in Kalman filtering.

Nowadays, more and more applications come out for conducting data fusion by applying the Kalman filter. Their accuracy requirements could be as high as a few cm in kinematic positioning, for example, in the modern direct georeferencing technology specifically. There is an urgent need to add the appropriate a posteriori precision assessment to its solution formulation. The overall objective of this manuscript aims how one may posteriorly improve the a priori stochastic model in terms of solution precision/accuracy evaluation with considering the implication of residuals of the predicted state vector, process and measurement noise vectors in Discrete Kalman filtering. Accordingly, this manuscript would systematically structure how to perform the a posteriori precision and accuracy assessment so that one can standardize its analytical formulation and also practical execution as an inalienable part of a Kalman filter.

This introduction is followed by Section 2 that summarizes the standard Kalman filter inclusive of system and measurement models along with the general assumption of stochastic model of process and measurement noise vectors, the core of the solution and its essential accompaniments to prepare for the theoretical basis. Section 3 moves on the main objective of this manuscript: a posteriori precision evaluation in discrete Kalman filtering. Then, a working example is presented from a real test data acquired by using a land-based GPS/IMU integrated kinematic positioning system in Section 4. At the end, Section 5 concludes the manuscript.

2. Standard Kalman Filter

2.1 System and Measurement Models

This section straightforward configures the formulation of a Kalman filter in discrete time that is referred to as the standard form through this manuscript.

Assume to have a linear state-space system with its discrete observation made over a time period of $(t_0, t_1, ..., t_k, ..., t_N)$, where each time instant implies an observation epoch. For simplification without loss of generality, the deterministic system input will be left out and the time instant t_k will also be substituted by k.

In general, at an observation epoch k, one has the system model

$$\boldsymbol{x}(k) = \boldsymbol{A}(k, k-1)\boldsymbol{x}(k-1) + \boldsymbol{B}(k, k-1)\boldsymbol{w}(k) \qquad (2.1)$$

wherein $\boldsymbol{x}(k), \boldsymbol{x}(k-1), \boldsymbol{w}(k)$ are the state vector at k, the state vector at k - 1, and the process noise vector, respectively while $\boldsymbol{A}(k, k-1)$ and $\boldsymbol{B}(k, k-1)$ are the system state and process noise transition matrices from observation epoch k – 1 to epoch k, respectively. For the purpose of predigesting the further analytic expressions, $\boldsymbol{A}(k, k-1)$ and $\boldsymbol{B}(k, k-1)$ are simplified to $\boldsymbol{A}(k)$ and $\boldsymbol{B}(k)$ wherever no confusion might be led to. In addition, the initial state vector is given as $\boldsymbol{x}(0)$ with its covariance matrix of $\boldsymbol{D}_{xx}(0)$.

The accompanying measurement model at the same observation epoch is given as

$$\boldsymbol{z}(k) = \boldsymbol{C}(k)\boldsymbol{x}(k) + \boldsymbol{\Delta}(k) \qquad (2.2)$$

wherein $\boldsymbol{z}(k), \boldsymbol{C}(k), \boldsymbol{\Delta}(k)$ are the measurement vector, the observable design matrix and the measurement

noise vector, respectively at k, respectively.

In an encouraging alternate way, Wang [1997] radically expressed (2.1) in two independent groups of pseudo-measurement vectors as follows:

$$\boldsymbol{l}_x(k) = \boldsymbol{A}(k,k-1)\boldsymbol{x}(k-1) = \boldsymbol{x}(k/k-1)$$
$$\boldsymbol{D}_{l_x l_x}(k) = \boldsymbol{A}(k,k-1)\boldsymbol{D}_{xx}(k-1)\boldsymbol{A}^T(k,k-1) \quad (2.3)$$

$$\boldsymbol{l}_w(k) = \boldsymbol{w}(k) \qquad \boldsymbol{D}_{l_w l_w}(k) = \boldsymbol{Q}(k) \quad (2.4)$$

wherein $\boldsymbol{l}_x(k)$, $\boldsymbol{l}_w(k)$ are the two pseudo measurement vectors from the purely predicted state vector and the process noise vector associated with their covariance matrices $\boldsymbol{D}_{l\,l_x}(k)$ and $\boldsymbol{D}_{l_w l_w}(k)$, respectively. By combining (2.2), (2.3) and (2.4) together with the stochastic model as in Section 2.2, the identical results were derived as in Section 2.3.1 and further delivered unique additional quantities for conducting a comprehensive error analysis in discrete Kalman filtering [Wang, 1997].

2.2 Stochastic Models

Three basic assumptions about the process and measurement noises are practically made in general.

First, the process noise vector satisfies

$$\boldsymbol{w}(k) \sim \boldsymbol{N}(\boldsymbol{o}, \boldsymbol{Q}(k)) \quad (2.5)$$

wherein $\boldsymbol{u} \sim \boldsymbol{N}(\boldsymbol{e}, \boldsymbol{V})$ reads that $\boldsymbol{u}$ (a variable or a vector) "conform" to the normal distribution with an expectation of $\boldsymbol{e}$ and covariance matrix of $\boldsymbol{V}$. (2.5) means that the process noise vector $\boldsymbol{w}(k)$ conforms to the normal distribution with its expectation of $\boldsymbol{o}$ (zero vector) and variance matrix of $\boldsymbol{Q}(k)$, in other words, $\boldsymbol{w}(k)$ is characterized as white noise with its variance matrix of $\boldsymbol{Q}(k)$

Second, the measurement noise vector $\boldsymbol{\Delta}(k)$ is attributed to

$$\boldsymbol{\Delta}(k) \sim \boldsymbol{N}(\boldsymbol{o}, \boldsymbol{R}(k)) \quad (2.6)$$

i.e., a normal distribution with its expectation of $\boldsymbol{o}$ (zero vector) and variance matrix of $\boldsymbol{R}(k)$.

Third, the process and measurement noises between two epochs i and j ($i \neq j$) are commonly characterized as independent to each other, that is,

$$Cov(\boldsymbol{w}(i), \boldsymbol{w}(j)) = \boldsymbol{O} \quad (2.7)$$

$$Cov(\boldsymbol{\Delta}(i), \boldsymbol{\Delta}(j)) = \boldsymbol{O} \quad (2.8)$$

$$Cov(\boldsymbol{w}(i), \boldsymbol{\Delta}(j)) = \boldsymbol{O} \quad (2.9)$$

and, furthermore, $\boldsymbol{w}(k)$ and $\boldsymbol{\Delta}(k)$ are also uncorrelated to the initial state vector:

$$Cov(\boldsymbol{w}(i), \boldsymbol{x}(0)) = \boldsymbol{O} \quad (2.10)$$

$$Cov(\boldsymbol{\Delta}(i), \boldsymbol{x}(0)) = \boldsymbol{O} \quad (2.11)$$

All the equations from (2.5) to (2.11) express the a-priori stochastic model and belong to a vitally necessary part of the standard formulation in discrete Kalman filtering.

2.3 The Solution

The epochwise solution for the state vector from the system and measurement models (Section 2.1) is commonly derived after the Principle of Minimum Variance [Gelb, 1974; etc.]. Under the assumption of normal distributions as in (2.5) and (2.6) plus (2.7) – (2.8), the solution after the Principle of Minimum Variance is undoubtedly identical to the solution after the Principle of Least-Squares [Wang, 1997; Wang et al, 2023].

2.3.1 The core of the solution

As the solution derivation is so well known, the core of the solution is directly summarized in Table 2.1 [Wang, 2009].

Table 2.1 The core of the solution in Kalman filtering

The time update (or the one step prediction)

$$\hat{\boldsymbol{x}}(k/k-1) = \boldsymbol{A}(k)\hat{\boldsymbol{x}}(k-1) \quad (2.12)$$

$$\boldsymbol{D}_{xx}(k/k-1) = \boldsymbol{A}(k)\boldsymbol{D}_{xx}(k-1)\boldsymbol{A}^T(k) + \boldsymbol{B}(k)\boldsymbol{Q}(k)\boldsymbol{B}^T(k) \quad (2.13)$$

The measurement update (or the optimal estimate)

$$\hat{\boldsymbol{x}}(k) = \hat{\boldsymbol{x}}(k/k-1) + \boldsymbol{G}(k)\boldsymbol{d}(k) \quad (2.14)$$

$$\boldsymbol{D}_{xx}(k) = \boldsymbol{G}(k)\boldsymbol{R}(k)\boldsymbol{G}^T(k) + [\boldsymbol{E} - \boldsymbol{G}(k)\boldsymbol{C}(k)]\boldsymbol{D}_{xx}(k/k-1)[\boldsymbol{E} - \boldsymbol{G}(k)\boldsymbol{C}(k)]^T \quad (2.15)$$

Note: $\boldsymbol{E}$ is the identity matrix.

The system innovation and gain matrix:

$$\boldsymbol{d}(k) = \boldsymbol{z}(k) - \boldsymbol{C}(k)\hat{\boldsymbol{x}}(k/k-1) \quad (2.16)$$

$$\boldsymbol{D}_{dd}(k) = \boldsymbol{C}(k)\boldsymbol{D}_{xx}(k/k-1)\boldsymbol{C}^T(k) + \boldsymbol{R}(k) \quad (2.17)$$

$$\boldsymbol{G}(k) = \boldsymbol{C}(k)\boldsymbol{D}_{xx}(k/k-1)\boldsymbol{C}^T(k)\boldsymbol{D}_{dd}^{-1}(k) \quad (2.18)$$

2.3.2 The essential accompaniments

In order to successfully deliver the solution as in Table 2.1, a number of the concomitant analytic analyses are run in parallel with the recursive time and measurement updates in Kalman filtering, which may include model optimization, statistic characterization of the process noises and measurement noises, and adaptive and/or robust measures. For example, the

characterization of the innovation series normally belongs to the key processes in Kalman filtering, so does the system fault detection. In line with the similar purpose, one demands for estimating and analyzing plenty of additional essential auxiliary quantities as necessary part of the data fusion process in Kalman filtering.

To avoid waste time on triviality, let come straight to the main theme. The objective is here laid on the comprehensive error analysis in discrete Kalman filtering [Wang, 1997, 2008, 2009; Wang et al, 2021]. By considering the three groups of the epochwise available uncorrelated stochastic information: (1) the process noise vector; (2) the purely predicted state vector (exclusive of the process noises) and (3) the measurement vector, Wang [1997] successfully introduced an alternate derivation that not only reached the identical solution as the one in Section 2.3.1 and also delivered certain essential accompaniments to the core solution, which are summarized below [Wang, 1997, 2008, 2009]:

1). The residuals of the process noise vector $\boldsymbol{w}(k)$

$$\boldsymbol{v}_w(k) = \boldsymbol{Q}(k)\boldsymbol{B}^T(k)\boldsymbol{D}_{xx}^{-1}(k/k-1)\boldsymbol{G}(k)\boldsymbol{d}(k) \quad (2.19)$$

$$\boldsymbol{D}_{v_w v_w}(k) = \boldsymbol{Q}(k)\boldsymbol{B}^T(k)\boldsymbol{C}^T(k)\boldsymbol{D}_{dd}^{-1}(k)\boldsymbol{C}(k)\boldsymbol{B}(k)\boldsymbol{Q}(k) \quad (2.20)$$

2). The residuals of the measurement vector $\boldsymbol{z}(k)$

$$\boldsymbol{v}_z(k) = [\boldsymbol{C}(k)\boldsymbol{G}(k) - \boldsymbol{E}]\boldsymbol{d}(k) \quad (2.21)$$

$$\boldsymbol{D}_{v_z v_z}(k) = [\boldsymbol{E} - \boldsymbol{C}(k)\boldsymbol{G}(k)]\boldsymbol{R}(k) \quad (2.22)$$

3). The residuals of the predicted state vector $\boldsymbol{l}_x(k)$

$$\boldsymbol{v}_{l_x}(k) = \boldsymbol{D}_{l_x l_x}(k)\boldsymbol{D}_{xx}^{-1}(k/k-1)\boldsymbol{G}(k)\boldsymbol{d}(k) \quad (2.23)$$

$$\boldsymbol{D}_{v_{l_x} v_{l_x}}(k) = \boldsymbol{D}_{l_x l_x}(k)\boldsymbol{C}^T(k)\boldsymbol{D}_{dd}^{-1}(k)\boldsymbol{C}(k)\boldsymbol{D}_{l_x l_x}(k) \quad (2.24)$$

wherein

$$\boldsymbol{l}_x(k) = A(k)\boldsymbol{x}(k-1/k-1) \quad (2.25)$$

$$\boldsymbol{D}_{l_x l_x}(k) = \boldsymbol{A}(k)\boldsymbol{D}_{xx}(k-1)\boldsymbol{A}^T(k) \quad (2.26)$$

4). The redundancy index of $\boldsymbol{w}(k)$

$$r_w(k) = trace\{\boldsymbol{Q}(k)\boldsymbol{B}^T(k)\boldsymbol{C}^T(k)\boldsymbol{D}_{dd}^{-1}(k)\boldsymbol{C}(k)\boldsymbol{B}(k)\} \quad (2.27)$$

which is the total redundancy contribution of the process noise vector. When $\boldsymbol{Q}(k)$ is diagonal, the individual redundancy index is given as

$$r_{w_i}(k) = \{\boldsymbol{Q}(k)\boldsymbol{B}^T(k)\boldsymbol{C}^T(k)\boldsymbol{D}_{dd}^{-1}(k)\boldsymbol{C}(k)\boldsymbol{B}(k)\}_{ii} \quad (2.28)$$

5). The redundancy index of $\boldsymbol{z}(k)$

$$r_z(k) = trace\{\boldsymbol{E} - \boldsymbol{C}(k)\boldsymbol{G}(k)\} \quad (2.29)$$

which is the total redundancy contribution of the measurement vector. When $\boldsymbol{R}(k)$ is diagonal, the individual redundancy index is given as

$$r_{z_i}(k) = 1 - \{\boldsymbol{C}(k)\boldsymbol{G}(k)\}_{ii} \quad (2.30)$$

6). The redundancy index of $\boldsymbol{l}_x(k)$

$$r_{l_x} = trace\{\boldsymbol{D}_{l_x l_x}(k)\boldsymbol{C}^T(k)\boldsymbol{D}_{dd}^{-1}(k)\boldsymbol{C}(k)\} \quad (2.31)$$

which is the total redundancy contribution of the predicted state vector. As the components in the predicted state vector $\boldsymbol{l}_x(k)$ are correlated to each other in general, their individual redundancy indexes are ordinarily not interpretable and therefore not adopted in practice.

7). The total redundancy of $\boldsymbol{w}(k)$, $\boldsymbol{z}(k)$ and $\boldsymbol{l}_x(k)$

$$r(k) = r_w(k) + r_z(k) + r_{l_x}(k) = p(k) \quad (2.32)$$

wherein $p(k)$ is the dimension of $\boldsymbol{z}(k)$, i.e., the total system redundancy is equal to the number of the measurements in Kalman filtering.

Wang, et al [2009] described three unique fundamental and practical usages of this redundancy contribution in Discrete Kalman filtering: (i) the degrees of freedom of test statistics [Wang, 1997, 2008], (ii) the simplified algorithm of variance component estimation [Wang. 1997, 2008; Caspary and Wang, 1998; Wang et al, 2009], and (iii) the a posteriori variance components based robust Kalman filter [Wang et al, 2010].

By the way, the authors kindly admit that the equations (2.21) and (2.22) became available from the very beginning as they could easily be deduced together with the core solution in Section 2.3.1. They are listed here just for completeness with all the residuals due to their necessity for the a posteriori precision or accuracy assessment in Section 3 of this manuscript. However, all the rest equations first appeared in [Wang, 1997]. Due to the traditional habitude, almost all the basic statistical analysis in Kalman filtering has been focused on the system innovation series, indeed [Mehra, 1970; Teunissen et al, 2021; etc.]. Statistical analysis of the process noise residuals (refer to (2.19) and (2.20)) has been scarce so far due to the lack of such analytic expression directly from the solution derivation after the Principle of Minimum Variance. All the equations from (2.19) to (2.32) as a whole provided for the fundamental of not only the comprehensive error analysis and also reliability theory in Discrete Kalman filtering [Wang, 1997, 2008, 2009 etc.], which were further extended to Kalman filter with constraints [Wang and Brunson, 2023].

3. A Posteriori Precision Evaluation in Discrete Kalman Filtering

This section first review how the precision of the estimated parameters has been posteriorly evaluated in the Method of Least Squares or more often called the Least Squares Adjustment in Geodesy and Surveying Engineering, and then proposes the similar framework with which the a posteriori precision can practically be undertaken in Discrete Kalman filtering. To avoid any confusion between precision and accuracy of an estimate after a specific optimal principle, the word, precision, is preferred here. As a matter of fact, it is well known that the precision and accuracy will merge in terms of the estimated states or parameters on the ground of the Method of Least Squares and the Principle of Minimum Variance as long as their unbiasedness holds and it is unnecessary to pursue their distinction here in this manuscript.

3.1 The State-of-Art A Posteriori Precision Evaluation in the Method of Least Squares

It is so matured how scientists and engineers posteriorly evaluate the precision of the least squares solutions. Helmert [1907] distinctly dwelt upon the topic of the a posteriori precision and accuracy evaluation repeatedly in his magnum opus of historic significance of "Die Ausgleichungsrechnung nach der Methode der Kleinsten Quadrte" (The Least Squares Adjustment), for example, (i) Section VI Schluβkontrolle (Final Control) in §7 under Chapter 3, (ii) Section III Mittlerer Fehler (standard error) in §2 under Chapter 4, and (iii) Section III Der mittlere Fehler der Gewichtseinheit (standard error of unit weight) in §2 under Chapter 4. A subsection of the a posteriori precision evaluation can also be found in Chapter IV Adjustment of Indirect Observations in [Wright and Hayford, 1906]. Obviously, the a posteriori precision evaluation of the estimated parameters has been not only based on the a priori stochastic models (especially the measurement weighting scheme), the geometry of the linear or linearized models, and also posteriorly grounded on the measurement residuals and the redundancy contribution in a linear or linearized system. Such a posteriori precision evaluation has consistently been one of the standard components in the method of least squares and can be found in all the widely used higher education textbooks [WCSM, 1959; Wells and Krakiwsky, 1971; Cross, 1983; Koch, 1987; Caspary 1988; Taylor, 1997; Rao and Toutenberg, 1999; Ghilani and Wolf, 2006; Kariya and Kurata, 2004; Wang et al, 2019; etc.].

In summary, the a posteriori precision evaluation of the estimated parameters has been standardized as an essential component in the method of least squares (or least squares adjustment) for a very long time (more than a century).

3.2 A Posteriori Precision Evaluation in Discrete Kalman filtering

3.2.1 Introduction

In Kalman filtering, the equation (2.15) has generally been regarded as the a posteriori covariance matrix of the optimally estimated state vector, for example, stated on page 173 in Farrell [2008]. One could not resist asking what impacts this so-called a posteriori covariance matrix here, indeed. In other words, what decides $\boldsymbol{D}_{xx}(k)$ in (2.15)? It does involve the system and measurement models given in (2.1) and (2.2) and also the a priori stochastic model from (2.5) to (2.11). But how about the a posteriori random errors presented by the residuals of the process and measurement noise vectors? Their impacts have never been integrated into (2.15) yet if readers seriously review those popular textbooks of the Kalman filter and the overwhelming relevant literature [Gelb, 1974; Anderson and Moore, 1979; Salzmann, 1988; Zarchan, 2005; Simon, 2006; Grewal and Andrews, 2008; Chui and Chen, 2009; Eubank, 2009; Brown and Hwang, 2012; Teunissen et al, 2021; etc.]. In contrast to the a posteriori precision evaluation in the method of least squares, the existence of this deficiency in Kalman filtering is indeed hard to believe as there have been plenty of the research activities to undertake a lot more complicated issues, instead of undertaking this fundamental aspect to enhance the core solution in Kalman filtering. However, we have to genuinely confess to this reality.

Fortunately and very encouragingly, a few of our antecessors in the field of Geodesy did initiate this very specific aspect [Pelzer, 1987; Yu et al, 1988; Tao, 1992; Koch, 1990]. Especially, a description of "Variance Factor Unknown" for the covariance matrix $\boldsymbol{Q}(k)$ of the process noise vector and the covariance matrix $\boldsymbol{R}(k)$ of the measurement vector was given in Section 318 Linear Dynamic Systems, Chapter 3 Models and Special Applications in the well-known book, Bayesian Inference with Geodetic Applications [Koch, 1990]. A relatively systematic studies of it have further been conducted comprehensively in [Wang, 1997, 2008, 2009; Wang, et al, 2009, 2010, 2021, 2023; Qian, et al, 2016; Qian, 2017].

Accordingly, this section enters upon proposing a practically feasible structure with its focus on the a posteriori estimation of variances at different level for achieving the a posteriori precision in discrete Kalman filtering.

3.2.2 Variance of Unit Weight

The variance of unit weight (also commonly called the variance factor or the reference variance denoted as σ_0^2 usually) characterizes the estimated variance of a hypothetical observation with unit weight when dealing with (real and pseudo) measurements of unequal accuracy [Helmert, 1907; Ghilani and Wolf, 2006; etc.]. It is one of the most essential quantities in statistical quality evaluation and hypothesis testing. On one hand, one can posteriorly estimate the variance factor by using the system innovations (refer to (2.16) and (2.17)) or the residuals (refer to (2.19) – (2.24). On the other hand, it can be estimated epochwise (locally), over a specific time interval (regionally) or across the whole dataset (globally).

The local variance of unit weight at an arbitrary instant t_k is estimated by using the system innovation vector in (2.16) and (2.17) [Pelzer, 1987; Tao, 1992]

$$\hat{\sigma}_{l0}^2(k) = \frac{\boldsymbol{d}^T(k)\boldsymbol{D}_{dd}^{-1}(k)\boldsymbol{d}(k)}{p(k)} \tag{3.1}$$

or the residuals from (2.19) to (2.24) [Wang, 1997]

$$\hat{\sigma}_{l0}^2(k) = \frac{\boldsymbol{v}_{l_x}^T(k)\boldsymbol{D}_{l_xl_x}^{-1}(k)\boldsymbol{v}_{l_x}(k) + \boldsymbol{v}_w^T(k)\boldsymbol{Q}^{-1}(k)\boldsymbol{v}_w(k) + \boldsymbol{v}_z^T(k)\boldsymbol{R}^{-1}(k)\boldsymbol{v}_z(k)}{p(k)} \tag{3.2}$$

wherein the subscript l stands for local or epochwise and the subscript 0 stands for the reference variance (same below). The equivalence between (3.1) and (3.2) was proofed by Wang [1997].

The regional variance of unit weight at an arbitrary instant t_k can be estimated over a time window, for example, $t_{k-j+1},...,t_k$ $(j>0)$ (with using j epochs together) in analog to (3.1) and (3.2):

$$\hat{\sigma}_{r0}^2(k) = \sum_{i=k-j+1}^{k}\boldsymbol{d}^T(i)\boldsymbol{D}_{dd}^{-1}(i)\boldsymbol{d}(i) \Big/ \sum_{i=k-j+1}^{k}p(i) \tag{3.3}$$

$$\hat{\sigma}_{r0}^2(k) = \frac{\sum_{i=k-j+1}^{k}\{\boldsymbol{v}_{l_x}^T(i)\boldsymbol{D}_{l_xl_x}^{-1}(i)\boldsymbol{v}_{l_x}(i) + \boldsymbol{v}_w^T(i)\boldsymbol{Q}^{-1}(i)\boldsymbol{v}_w(i) + \boldsymbol{v}_z^T(i)\boldsymbol{R}^{-1}(i)\boldsymbol{v}_z(i))\}}{\sum_{i=k-j+1}^{k}p(i)} \tag{3.4}$$

by using the system innovation and residual series over the specified time window, respectively. The subscript r in (3.3) and (3.4) stands for regional. (3.3) and (3.4) are equivalent and deliver a more stable average value than (3.1) and (3.2) as the epochwise estimated variance factors may wander around from time to time. The validity of (3.3) and (3.4) hold due to the fact that the system innovation and residuals are independent between different epochs under the given stochastic models in Section 2.2 [Tao, 1992; Wang, 1997].

The global variance of unit weight can be further estimated across the whole data over $(t_1,...,t_k,...,t_N)$:

$$\hat{\sigma}_{g0}^2 = \sum_{i=1}^{N}\boldsymbol{d}^T(i)\boldsymbol{D}_{dd}^{-1}(i)\boldsymbol{d}(i) \Big/ \sum_{i=1}^{N}p(i) \tag{3.5}$$

$$\hat{\sigma}_{g0}^2 = \frac{\sum_{i=1}^{N}\{\boldsymbol{v}_{l_x}^T(i)\boldsymbol{D}_{l_xl_x}^{-1}(i)\boldsymbol{v}_{l_x}(i) + \boldsymbol{v}_w^T(i)\boldsymbol{Q}^{-1}(i)\boldsymbol{v}_w(i) + \boldsymbol{v}_z^T(i)\boldsymbol{R}^{-1}(i)\boldsymbol{v}_z(i)\}}{\sum_{i=1}^{N}p(i)} \tag{3.6}$$

wherein the subscript g in (3.5) and (3.6) stands for global. As long as the a priori stochastic models, i.e., the covariance matrices, $\boldsymbol{R}(1),\boldsymbol{R}(2),...,\boldsymbol{R}(k)$ and $\boldsymbol{Q}(1),\boldsymbol{Q}(2),...,\boldsymbol{Q}(k)$ along with $\boldsymbol{D}_{x_0x_0}(0)$ stochastically well characterize the system as defined in Section 2.1, the estimated a posteriori variance factors should sufficiently close to the Unity. Otherwise, one may consider posteriorly scale $\boldsymbol{D}_{xx}(k)$ using an estimated variance factor, $\hat{\sigma}_0^2$, as follows:

$$\hat{\boldsymbol{D}}_{xx}(k) = \hat{\sigma}_0^2\boldsymbol{D}_{xx}(k) \tag{3.7}$$

which represents the a posteriori covariance matrix of $\hat{\boldsymbol{x}}(k)$ with having the influence of the used data via $\hat{\sigma}_0^2$. Unfortunately, the lack of this a posteriori measure has been widespread indeed.

The estimate after (3.5) and (3.6) is preferred in post processing because it provides an overall evaluation in terms of the variance factor as a scale in its magnitude to reflect the average noise level of the entire system solution so that one may scale the given a priori covariance matrices and then iterate the data fusion until its estimate statistically converges to the unity. The estimate after (3.1) and (3.2) can be utilized to

effectively conduct system diagonals for any fault related to the system model and/or the measurement model. One may use the estimate after (3.3) and (3.4) to identify any data gaps in measurements, any over parameterization, or modeling deficiency. In brief, how one appropriately utilizes them is closely related to the applications and analysts' understanding of the being processed data.

3.2.3 Separate Variance Factors for Q and R

What about if the a priori covariance matrix $\boldsymbol{Q}(k)$ of the process noise vector $\boldsymbol{w}(k)$ does not share the same variance factor with the a priori covariance matrix $\boldsymbol{R}(k)$ of the measurement vector $\boldsymbol{z}(k)$? In connection with the given question, this section seeks for a practically feasible solution.

In Kalman filtering, the process noise vector and the measurement noise vector could straightforwardly be simply treated as two uncorrelated types of the observation (or pseudo observations) so that they are associated with their own stochastic models. Besides Section 3.2.1, one can further estimate the variance factors for $\boldsymbol{Q}$ and $\boldsymbol{R}$ simultaneously. The variance component estimation (VCE) method after Helmert in the method of least squares [Welsch, 1978; Förstner, 1979; Koch, 1987; Cui et al, 1993; etc.], for example, can be deployed for such purpose. For practical purpose, Förstner [1979] proposed a simplified algorithm directly based on the measurement residuals and their redundancy contribution, which has been popularized in aerial photogrammetry where the number of the redundant measurements is large. By taking the advantage of high accumulative redundancy in Kalman filtering, Wang [1997] successfully realized Förstner's simplified algorithm, by which the given question at the start can be resolved. In analog to (3.2), (3.4) and (3.6), from (2.19) - (2.22) and (2.27) and (2.29), the local, regional and global variance factors are posteriorly estimated for $\boldsymbol{Q}$ (with the subscript of Q)

$$\hat{\sigma}^2_{l0Q}(k) = \frac{\boldsymbol{v}_w^T(k)\boldsymbol{Q}^{-1}(k)\boldsymbol{v}_w(k)}{r_w(k)} \qquad (\text{at } t_k) \tag{3.8}$$

$$\hat{\sigma}^2_{r0Q}(k) = \frac{\sum_{i=k-j+1}^{k} \boldsymbol{v}_w^T(i)\boldsymbol{Q}^{-1}(i)\boldsymbol{v}_w(i)}{\sum_{i=k-j+1}^{k} r_w(i)} \quad (\text{over } [t_{k-j+1}, t_k]) \tag{3.9}$$

$$\hat{\sigma}^2_{g0Q}(k) = \frac{\sum_{i=1}^{N} \boldsymbol{v}_w^T(i)\boldsymbol{Q}^{-1}(i)\boldsymbol{v}_w(i)}{\sum_{i=1}^{N} r_w(i)} \quad (\text{across } [t_1, t_N]) \tag{3.10}$$

and for $\boldsymbol{R}$ (with the subscript of R)

$$\hat{\sigma}^2_{l0R}(k) = \frac{\boldsymbol{v}_z^T(k)\boldsymbol{R}^{-1}(k)\boldsymbol{v}_z(k)}{r_z(k)} \qquad (\text{at } t_k) \tag{3.11}$$

$$\hat{\sigma}^2_{r0R}(k) = \frac{\sum_{i=k-j+1}^{k} \boldsymbol{v}_z^T(i)\boldsymbol{R}^{-1}(i)\boldsymbol{v}_z(i)}{\sum_{i=k-j+1}^{k} r_z(i)} \quad (\text{over } [t_{k-j+1}, t_k]) \tag{3.12}$$

$$\hat{\sigma}^2_{g0R} = \frac{\sum_{i=1}^{N} \boldsymbol{v}_z^T(i)\boldsymbol{R}^{-1}(i)\boldsymbol{v}_z(i)}{\sum_{i=1}^{N} r_z(i)} \quad (\text{across } [t_1, t_N]) \tag{3.13}$$

Thus, one can choose specific estimates of these two variance factors to scale the a priori process and measurement noise matrices as follows:

$$\hat{\boldsymbol{Q}}(k) = \hat{\sigma}^2_{0Q}\boldsymbol{Q}(k) \tag{3.14}$$

$$\hat{\boldsymbol{R}}(k) = \hat{\sigma}^2_{0R}\boldsymbol{R}(k) \tag{3.15}$$

which are called the a posteriori covariance matrices of the process and measurement noise vectors, respectively.

3.2.4 Individual Variance Factors for the uncorrelated groups of process noise factors and measurements

There exist plenty of applications that may need to posteriorly estimate the variance factors of different types of measurements or independent measurement groups. A typical example is the relative GNSS kinematic positioning of a moving rover with respect to a stationary receiver at a base station. Specifically consider using GPS receivers with three independent measurement types: L1 C/A, L1 carrier phases and L2 carrier phases. So, one may need to estimate three variance factors corresponding to the double-differenced (DD) L1 C/A, L1 carrier phases and L2 carrier phases. Due to the double differencing process, the DD measurements are correlated within each type [Gopaul et al, 2010; Wang et al 2010]. With the progress of the GPS modernization program, there could be six uncorrelated types of GPS measurements with the range and phase measurements from L1, L2 and L5. Another example is that an IMU could acquire a 3 dimensional specific force vector and a 3 dimensional angular rate vector so that two variance factors may be introduced in their variance component estimation for each, respectively [Wang et al, 2021], even more variance factors for an IMU array [Brunson et al, 2024].

Without loss of generality, one no more needs to distinguish between the process noise vector and measurement noise vector in this subsection. Instead, the variance component (or factor) estimation is

focused on the grouped measurements no matter there exists or does not exist any correlation within a group, but no correlation exists between measurement groups.

Let $\boldsymbol{v}_s(k) \sim N(o, \boldsymbol{D}_{ss}(k))$ at instant t_k, where $\boldsymbol{v}_s(k)$ is the residual vector of the *s*-th group (the partial tone from ***s***et) of the measurement vector with its a priori covariance matrix $\boldsymbol{D}_{ss}(k)$. Similar to (2.27), (2.29) and (2.31), its $r_s(k)$ stand for the subtotal redundancy contribution of the *s*-th group of the measurements. Correspondingly, the variance factor for the *s*-th group is estimated as follows:

$$\hat{\sigma}^2_{l0/s}(k) = \frac{\boldsymbol{v}_s^T(k)\boldsymbol{D}_{ss}^{-1}(k)\boldsymbol{v}_s(k)}{r_s(k)} \quad (\text{at } t_k) \tag{3.16}$$

$$\hat{\sigma}^2_{r0/s}(k) = \frac{\sum_{i=k-j+1}^{k} \boldsymbol{v}_s^T(i)\boldsymbol{D}_{ss}^{-1}(i)\boldsymbol{v}_s(i)}{\sum_{i=k-j+1}^{k} r_s(i)} \quad (\text{over } [t_{k-j+1}, t_k]) \tag{3.17}$$

$$\hat{\sigma}^2_{g0/s} = \frac{\sum_{i=1}^{N} \boldsymbol{v}_s^T(i)\boldsymbol{D}_{ss}^{-1}(i)\boldsymbol{v}_s(i)}{\sum_{i=1}^{N} r_s(i)} \quad (\text{across } [t_1, t_N]) \tag{3.18}$$

It must be pointed out that the degrees of freedom are the total redundancy contribution of all the measurements, instead of simply the number of the measurements in the *s*-th group, which is generally smaller than the latter. Clearly, such variance factors cannot be estimated directly through using the system innovation series.

In general, the a posteriori estimate of the a priori covariance matrix $\boldsymbol{D}_{ss}(k)$ is given as follows:

$$\hat{\boldsymbol{D}}_{ss}(k) = \hat{\sigma}^2_{0/s}\boldsymbol{D}_{ss}(k) \tag{3.19}$$

for the *s*-th group of the measurements inclusive of the specific groups of $\boldsymbol{w}(k)$ in (2.1)and/or $\boldsymbol{z}(k)$ in (2.2), wherein $\hat{\sigma}^2_{0/s}$ is a chosen variance factor among (3.16) – (3.18).

3.2.5 Individual Variance Factors or Components for the uncorrelated process noise factors and measurements

In comparison with the scenario as described in Section 3.2.4, one can further estimate the variance components for each of the uncorrelated process noise factors and/or uncorrelated measurement types, which can be completed either through estimating their variance factors or variance components at a level further in detail.

Commonly, the individual components in a process noise vector are assumed to be a priori independent of each other, i.e., the covariance matrix $\boldsymbol{Q}(k)$ in (2.5) is diagonal in practice unless otherwise stated. Quite the same with the measurements from different sensors in a multi-sensor kinematic positioning and navigation system, the different measurements at a time are presumed to be uncorrelated to each other, for example, the measurements from the three gyros and three accelerometers in an IMU. So, they can be treated as six independent measurements. Another example is the standard GPS single point positioning using the L1 C/A pseudoranges, where one can estimate the variance components associated with each satellite [Wang, et al, 2009; Gopaul et al, 2010].

Under the assumption that $\boldsymbol{Q}(k)$ in (2.5) and $\boldsymbol{R}(k)$ in (2.6) (k=1, 2, …, i, …, N) are diagonal or partially diagonal, the individual variance factors for their uncorrelated components can posteriorly be estimated as follows:

$$\hat{\sigma}^2_{l0\boldsymbol{w}_i}(k) = \frac{v^2_{\boldsymbol{w}_i}(k)/Q_{\boldsymbol{w}_i}(k)}{r_{\boldsymbol{w}_i}(k)} \quad (\text{at } t_k) \tag{3.20}$$

$$\hat{\sigma}^2_{r0w_i}(k) = \frac{\sum_{u=k-j+1}^{k} v^2_{w_i}(u)/Q_{w_i}(u)}{\sum_{u=k-j+1}^{k} r_{w_i}(u)} \quad (\text{over } [t_{k-j+1}, t_k]) \tag{3.21}$$

$$\hat{\sigma}^2_{g0w_i} = \frac{\sum_{u=1}^{N} v^2_{w_i}(u)/Q_{w_i}(u)}{\sum_{u=1}^{N} r_{w_i}(u)} \quad (\text{across } [t_1, t_N]) \tag{3.22}$$

for $\boldsymbol{Q}(k)$ after (2.19) and (2.28), and

$$\hat{\sigma}^2_{l0z_i}(k) = \frac{v^2_{z_i}(k)/R_{z_i}(k)}{r_{z_i}(k)} \quad (\text{at } t_k) \tag{3.23}$$

$$\hat{\sigma}^2_{r0z_i}(k) = \frac{\sum_{u=k-j+1}^{k} v^2_{z_i}(u)/R_{z_i}(u)}{\sum_{u=k-j+1}^{k} r_{z_i}(u)} \quad (\text{over } [t_{k-j+1}, t_k]) \tag{3.24}$$

$$\hat{\sigma}^2_{g0z_i} = \frac{\sum_{u=1}^{N} v^2_{z_i}(u)/R_{z_i}(u)}{\sum_{u=1}^{N} r_{z_i}(u)} \quad (\text{across } [t_1, t_N]) \tag{3.25}$$

for $\boldsymbol{R}(k)$ after (2.21) and (2.29), which could be used to scale their a priori variances.

In case that $\boldsymbol{Q} = \boldsymbol{Q}(1) = \boldsymbol{Q}(2) = ... = \boldsymbol{Q}(N)$ and $\boldsymbol{R} = \boldsymbol{R}(1) = \boldsymbol{R}(2) = ... = \boldsymbol{R}(N)$, i.e., constant in addition to diagonal, the individual variance components for their uncorrelated components can posteriorly be estimated as follows:

$$\hat{\sigma}^2_{w_i}(k) = \frac{v^2_{w_i}(k)}{r_{w_i}(k)} \quad \text{(at } t_k\text{)} \tag{3.26}$$

$$\hat{\sigma}^2_{rw_i}(k) = \frac{\sum_{u=k-j+1}^{k} v^2_{w_i}(u)}{\sum_{u=k-j+1}^{k} r_{w_i}(u)} \quad \text{(over } [t_{k-j+1}, t_k]\text{)} \tag{3.27}$$

$$\hat{\sigma}^2_{gw_i} = \frac{\sum_{u=1}^{N} v^2_{w_i}(u)}{\sum_{u=1}^{N} r_{w_i}(u)} \quad \text{(across the data } [t_1, t_N]\text{)} \tag{3.28}$$

for $\boldsymbol{Q}(k)$ and

$$\hat{\sigma}^2_{z_i}(k) = \frac{v^2_{z_i}(k)}{r_{z_i}(k)} \quad \text{(at } t_k\text{)} \tag{3.29}$$

$$\hat{\sigma}^2_{rz_i}(k) = \frac{\sum_{u=k-j+1}^{k} v^2_{z_i}(u)}{\sum_{u=k-j+1}^{k} r_{z_i}(u)} \quad \text{(over } [t_{k-j\_1}, t_k]\text{)} \tag{3.30}$$

$$\hat{\sigma}^2_{gz_i} = \sum_{u=1}^{N} \frac{v^2_{z_i}(u)}{r_{z_i}(u)} \quad \text{(across } [t_1, t_N]\text{)} \tag{3.31}$$

for $\boldsymbol{R}(k)$, which could be used to update their a priori variances.

It is worth mentioning that with the VCE of the independent individual components in the process noise vector or in the measurement vector, some of the corresponding redundancy indexes may relatively small, especially with some of the process noise factors [Wang, et al, 2009]. This could result in verging on divergence of the estimated variance while its redundancy contribution becomes increasingly small in case the specific variance becomes very small (close to a high leverage random variable as in linear regression). To avoid a potential divergence of this type of variance components, one can simply exclude it from the VCE process by fixing its variance value (or reasonably adapt its a priori value).

3.3 General Tactics for how to proceed with the A Posteriori Precision Evaluation

Frankly, one can utilize the proposed four levels of a posteriori precision evaluation in many conceivable combinations in practice, particularly in post processing. On one hand, one needs to take all factors into consideration for achieving their best usage. On the other hand, the understanding and experience of a specific analyst do play important role. The authors hope that readers may be able to convert our following general view into actionable goals in their practice.

The usage of global variance of unit weight in (3.5) and (3.6) had better first be considered in order to fit your system states to the overall quality of the measurements. The regional and local variances of unit weight may provide better information about any regional and local characteristics, anomalies or noise homogeneities for conducting outlier detection or being concerned with the solution robustness and/or adaptivity.

Often, a user may not know the quantitative characteristic of the process noises as good as the ones of the measurement noises because one can easily refer to the technical specifications associated with the sensors for their reference (or nominal) values. In this case, one may hardly expect that the a priori covariance matrix $\boldsymbol{Q}$ for the process noise vector and the a priori covariance matrix $\boldsymbol{R}$ for the measurement noise vector at an epoch may share the same variance factor of unit weight, i.e., the reference variance. Hence, the a posteriori estimation of the separate variance factors for $\boldsymbol{Q}$ and R in Section 3.2.5 should be considered because the whole system through its integration of the measurements can adjust the absolute noise levels for both of the process and measurement noises together simultaneously.

In general, different sensors offer different types of measurements. Sometime, a single sensor may also offer multiple types of measurements. For example, an IMU offers a three dimensional specific force vector and a three dimensional angular rate vector at an observation epoch, which can be considered as two independent groups of measurements. So, one can posteriorly estimate their own variance factors after Section 3.2.4. Because the three components in either the specific force vector or the angular rate vector are also uncorrelated, one can surely consider having six independent groups of measurements and performing variance component estimation for each of them (six variance components in total) after Section 3.2.5. Moreover, let take a look into GPS relative kinematic positioning under the consideration of three independent types (or groups) of the measurements, i.e., L1 C/A (pseudoranges) and L1/L2 carrier phases associated with each available satellite. Although the raw measurements are even uncorrelated in each group from time to time, the double differenced (DD) measurements become correlated within each group. However, the DD GPS measurements are still independent to each other between groups. In this case, three variance factors can be posteriorly estimated for each group, but an estimation of any variance components for specific groups of satellites, for instance, per elevation angles or per individual satellites becomes meaningless. Besides, a unique type

of a posteriori variance based robust and adaptive Kalman filter was once proposed in kinematic positioning by Wang et al [2010].

While the schemes in Sections 3.2.2 and 3.2.3 are undertaken for practically improving of the a priori stochastic models in terms of systems, the schemes described in Sections 3.2.4 and 3.2.5 possess high potential for labeling or confirming the variances of specific measurement types, especially in the so-called online sensor calibration, especially with using high quality of carefully planned GNSS observations for calibrating other sensors. One can definitely benefit all of the proposed schemes for the a posteriori precision evaluation in post processing. But at the same time, the authors are nor suggesting that no benefit may be gained from them in real time. The outlined local and regional measures may be taken in real time mode correspondingly.

Ultimately, it is worth noting that the a posteriori precision estimation is to improve the a priori stochastic model based on the being processed real data instead of guaranteeing a higher precision or accuracy.

In addition, the authors feel obliged to further comment on the equations for variance factors or components in Section 3.2.3 – Section 3.2.5. They are simplified from the rigorous results after Helmert method [Förstner, 1979; Cui et al, 1993; Li and Yuan, 2002; Wang et al, 2009]. Their approximatability relies on the large number of redundant measurements, i.e., the total redundancy contribution of a system [Förstner, 1979; Li and Yuan, 2002; etc.]. A typical successful application is its usage in aerial photogrammetry [Förstner, 1979; Li and Yuan, 2002; etc.]. Wang et al [2009] specifically studied its approximatability vs. the rigorous method after Helmert in GNSS kinematic positioning and showed its choiceness in practice. Indeed, there could be sufficient number of measurements from a multi-sensor integrated kinematic positioning and navigation mission. For instance, a relative GNSS positioning mission can have 5400 measurements for 30 minutes at 1Hz data rate from 8 satellites using a rover relative to a base station. In GNSS/IMU integrated direct georeferencing system, one can even have 1,080,000 IMU measurements at 100 Hz data rate for 30 minutes. In comparison with our traditional geodetic control network, even in photogrammetry using Least squares adjustment, an appropriate utilization of such large number of the (redundant) measurements and their residuals can comfortably result in the more reliable variance component estimation. With using such large number of the measurements, especially the sufficiently large number of the redundant measurements, one can well characterize the process and measurement noises.

4. Working Examples and Discussions

This section illustrates the utility of *a posteriori* precision evaluation in Kalman filtering using a road test dataset.

4.1 Road Dataset

The data was acquired using our in-house developed kinematic positioning and navigation system consisting of a Vectornav VN-100 IMU operating at 100 Hz and a high-rate NovAtel OEM6 receiver mounted on a land vehicle with a second NovAtel OEM6 receiver acting as a base station at a fixed position.

The vehicle was driven in circles about a residential court for 5 minutes after a stationary start for the first 5 minutes, then stayed stationary for5 more minutes, and was further driven in kinematic for approximately 30 minutes. The top view of the trajectory is shown in Figure 4.1, while its velocity and acceleration profiles are shown in Figures 4.2 and 4.3, respectively.

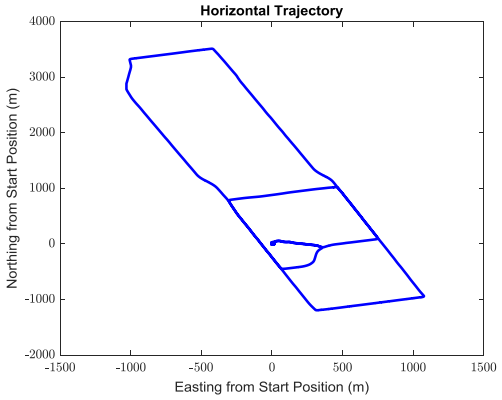

Figure 4.1: The top view of the trajectory (the coordinates are presented as local geodetic coordinates with respect to the starting location)

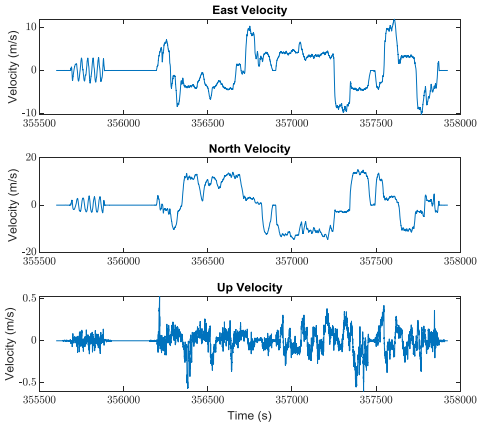

Figure 4.2: The Velocity profile of the kinematic trajectory

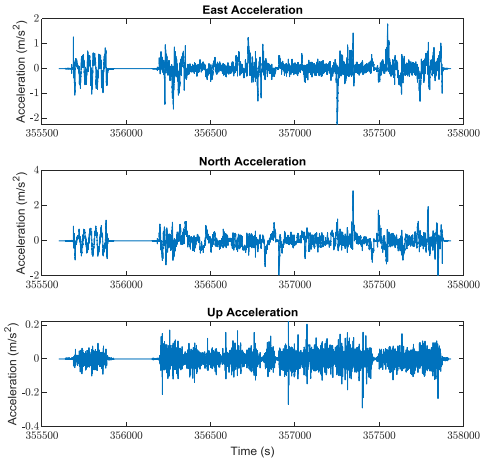

Figure 4.3: The acceleration profile of the kinematic trajectory

The attitude profile of the vehicle is shown in Figure 4.4, and its attitude rate-of-change is shown in Figure 4.5.

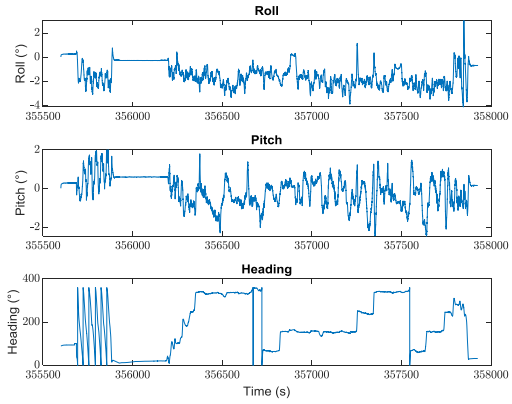

Figure 4.4: The attitude profile of the kinematic trajectory

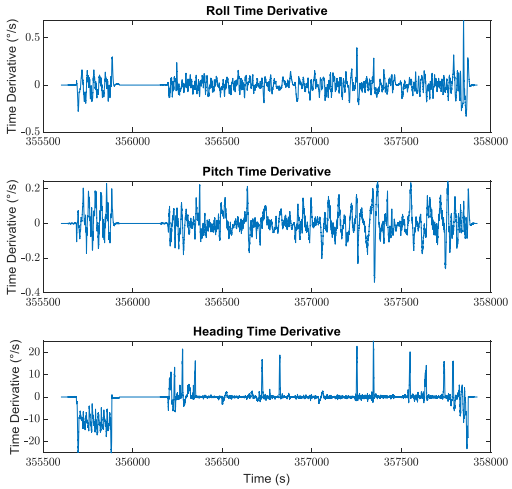

Figure 4.5: The attitude change profile of the kinematic trajectory

The *a priori* observation standard errors used in Kalman filtering are listed in Table 4.1.

Table 4.1: The a priori observation standard errors used in data processing

Observation Type	*A Priori* Standard Error
L1 C/A	±50 cm
L1 Carrier Phase	±5 mm
L2 Carrier Phase	±5 mm
Accelerometer	±10 cm/s^2
Gyroscope	±40’/s

The *a priori* standard errors of the process noises used in this example are summarized in Table.

Table 4.2: The a priori standard errors of the process noises used in data processing

Process Noise Components	*A Priori* Standard Error
Jerks	±10 m/s^3
The 2nd order of Attitude Derivatives	±1°/s^2

4.2 Overview of the Kalman filter

The data processing of this working example uses an identical approach to the Roll-Pitch-Heading model to define the Kalman Filter as in [Brunson and Wang, 2023]. Since the focus here is on the *posteriori* precision evaluation, special attention is paid to the definition of the ***R*** and ***Q*** matrices.

As in [Brunson and Wang, 2023], the state vector includes the linear position, velocity and acceleration vectors, as well as the roll, pitch and heading and their associated time-derivatives. The system model is constructed using a constant acceleration and constant attitude first-order time-derivative model. The state vector also includes any float estimates of the double-differenced GNSS ambiguity estimates, as well as the IMU accelerometer/gyroscope biases and scale factor errors. The system model for these additional state vector elements is constructed using a random-constant model.

The Kalman Filter used here is constructed after the Generic Multisensor Integration Strategy (GMIS) as in [Brunson and Wang, 2023]. One of the key differences between the GMIS and the Traditional Multisensor Integration Strategy (TMIS) is that the GMIS models all IMU outputs as observations in the Kalman Filter. This allows for the direct inclusion of IMU biases and scale factor errors in the state vector, and additionally allows for VCE to characterize the performance of the IMU in GNSS/IMU integrated systems.

Given the definition of the state vector, the process noise vector consists of the jerk (i.e. third-order position time-derivative) and the attitude second-order time-derivative vectors. The process noise vector also includes those factors relating to the shaping filters for estimating the IMU systematic errors and GNSS float ambiguity estimates, although they are not the focus of this analysis. The process noise vector is partitioned as follows

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{w}_p^T & \boldsymbol{w}_a^T & \boldsymbol{w}_\lambda^T & \boldsymbol{w}_{IMU}^T \end{bmatrix}^T \tag{4.1}$$

where $\boldsymbol{w_p}$ is the system jerk; $\boldsymbol{w_a}$ is the vector of second time-derivatives of the attitude parameters; $\boldsymbol{w_\lambda}$ describes the first-order time derivatives of the GNSS float ambiguity estimates; and $\boldsymbol{w_{IMU}}$ describes the first-order time derivatives of the gyroscope/accelerometer bias and scale factor error estimates.

The corresponding covariance matrix for the process noise vector is defined as

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q_{w_p}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q_{w_a}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Q_{w_\lambda}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Q_{w_{IMU}}} \end{bmatrix} \tag{4.2}$$

where the individual diagonal elements of $\boldsymbol{Q_{w_p}}$ and $\boldsymbol{Q_{w_a}}$ are defined after Table 4.2. Meanwhile, the diagonal elements of $\boldsymbol{Q_{w_\lambda}}$ and $\boldsymbol{Q_{w_{IMU}}}$ are quite small, since these quantities are not expected to drift quickly.

The observation vector at epoch *k* consist of three gyroscope observations, three accelerometer observations, the double differenced L1 C/A, and L1 and L2 Carrier Phase observations. The corresponding covariance matrix for the overall observation vector may therefore be defined to be

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{R_g} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R_s} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R_{GPS,C}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R_{GPS,L1}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R_{GPS,L2}} \end{bmatrix} \tag{4.3}$$

where $\boldsymbol{R_g}$ and $\boldsymbol{R_s}$ are 3x3 diagonal matrices describing the accuracy of the gyroscope and accelerometer outputs, respectively, $\boldsymbol{R_{GPS,C}}$ models the covariance matrix of the double-differenced L1 C/A observations, and $\boldsymbol{R_{GPS,L1}}$ and $\boldsymbol{R_{GPS,L2}}$ model the covariance matrices of the L1 and L2 Carrier Phase observations, respectively.

It is important to bear in mind that $\boldsymbol{R_g}$ and $\boldsymbol{R_s}$ are both diagonal matrices, since each gyroscope/accelerometer axis is assumed to be statistically independent. $\boldsymbol{R_{GPS,C}}$, $\boldsymbol{R_{GPS,L1}}$ and $\boldsymbol{R_{GPS,L2}}$ are fully populated, since all double-differenced GPS observations become correlated due to the use of a base satellite in the double differencing process at each epoch. Practically, this means that separate variance components could be used to characterize each axis of an IMU in a very straightforward manner, but that doing this sort of analysis for individual double-differenced GPS observations is a much more involved process.

4.3 The *A Posteriori* Evaluation of Observation and Process Noise Covariance Matrices

The appropriate tuning of the observation and process noise covariance matrices is a critical task in any kinematic positioning application. This is often initially determined using instrument specifications, but there are several factors that are generally unaccounted for in these *a priori* accuracy estimates:

i. Instrument accuracies are typically evaluated in a controlled laboratory environment, and this level of accuracy is rarely achieved in a complex real-world environment.
ii. There are many environmental effects that can significantly degrade the quality of signals from a particular positioning sensor. Examples include multipath errors in GNSS observations and the effects of vibration on IMU observations.

Properly accounting for these factors is complicated and in practice, there can be a lot of guesswork involved in this process. The a *posteriori* precision evaluation can provide valuable tools for refining the observation and process noise covariance matrix in a post-processing environment.

The standard error factors for each of the observation types and process noise components were evaluated over the duration of the kinematic dataset and used to iteratively scale the *a priori* standard errors. Once the standard error factors all converged to approximately 1, the covariance matrices were considered well-tuned within the system. Here, it took 5 iterations for the tuning of the observation and process noise covariance matrices to converge, and Figures 4.6 and 4.7 illustrate this process.

Figure 4.6: The standard error factors estimated for each observation type as part of the iterative tuning of the observation covariance matrix **R**.

After the final tuning of the observation and process noise covariance matrices, the estimated *a posteriori* standard errors are summarized in Table for the observations and Table for the process noise.

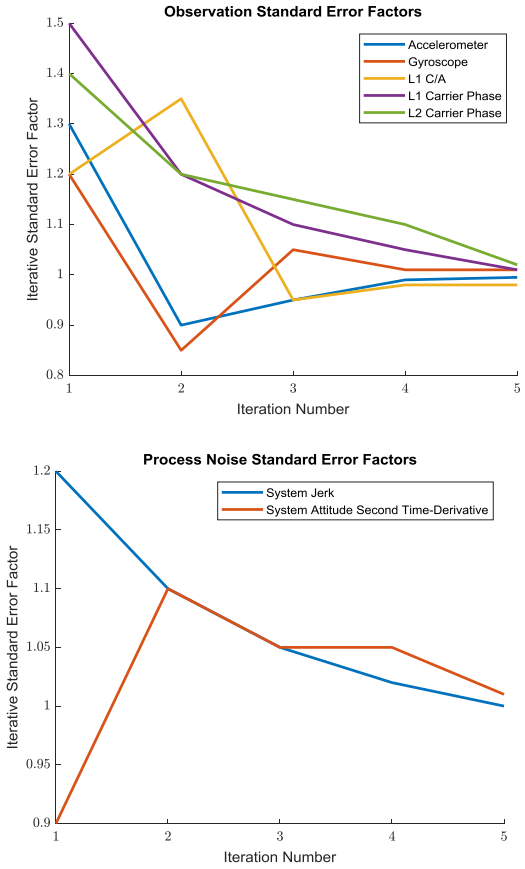

Figure 4.7: The a posteriori standard error factors for each type of the process noises as part of the iterative tuning of the process noise covariance matrix.

Table 4.3: The a posteriori observation standard errors.

Observation Type	*A Posteriori* Standard Error
L1 C/A	±74 cm
L1 Carrier Phase	±10 mm
L2 Carrier Phase	±11 mm
Accelerometer	±11 cm/s^2
Gyroscope	±45 ‘/s

Table 4.4: The a posteriori process noise standard errors.

Process Noise Components	*A Posteriori* Standard Error
Jerk	±14 m/s^3
The 2nd order of Attitude Derivatives	±1.1 °/s^2

4.4 Time-Varying Standard Error Factor Estimates

Section 4.2 focuses on using the estimated observation and process noise residuals to determine overall (or global) standard error factors. This section instead focuses on time-varying estimates of the standard error factors for different observation types.

It is possible to estimate epoch-wise (local) standard error factors for the observation and process noise vectors using their residuals and redundancy contributions as defined in Section 2.3.2. It is necessary to understand the limitations of doing this, since redundancy contributions may be very small for a given epoch, particularly for the process noise vector (Figure 4.8). To illustrate this, we present the redundancy contributions of the observation, process noise, and system state vectors in Figure 4.8. It is clear that the observation vector makes the most dominant contribution to the system’s overall redundancy. Since the redundancy contribution of the process noise vector to the system is relatively small for a given epoch, the epoch-wise estimates of the process noise standard error factors will be unreliable.

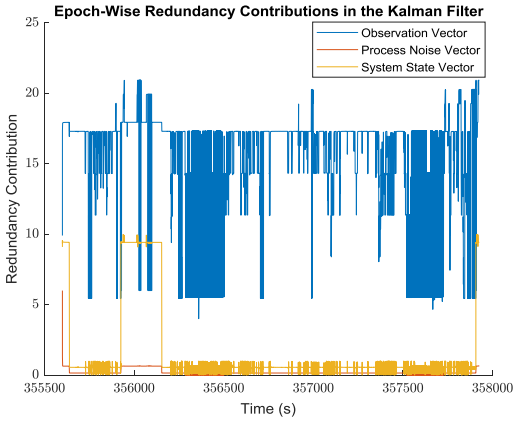

Figure 4.8: The epoch-wise redundancy contributions of the observation, process noise, and predicted system state vectors over the kinematic dataset (Note: the sum of all three redundancy contributions is equal to the number of observations plus the number of state constraints)

To counter this issue, the standard error factors may be estimated using a moving window (regional) to reduce the time-resolution of the standard error factor estimates, but significantly improve their reliability. See Figure for an illustration of the effects of different window sizes on the estimated standard error factors for the overall observation vector. Figure shows the effects of different window sizes on the standard error factors for the process noise vector.

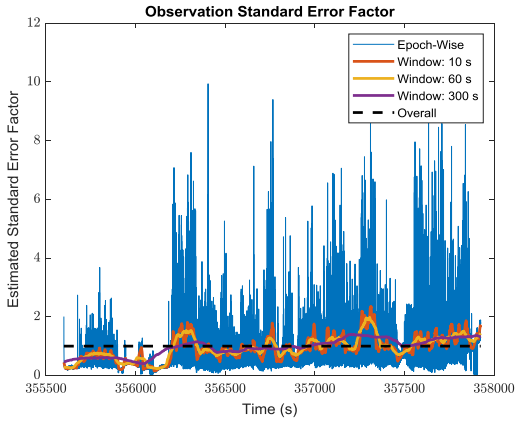

Figure 4.9: An illustration of the effects of different window sizes on the observation standard error factors.

Doubtless, the standard error factors may be estimated for the different observation groups: L1 C/A (Figure 4.11), L1 Carrier Phase (Figure 4.12), L2 Carrier Phase (Figure 4.13), gyroscopes (Figure 4.14), and accelerometers (Figure 4.15).

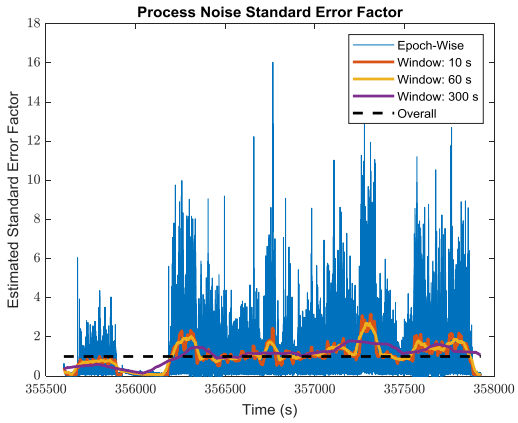

Figure 4.10: An illustration of the effects of different window sizes on the process noise standard error factors.

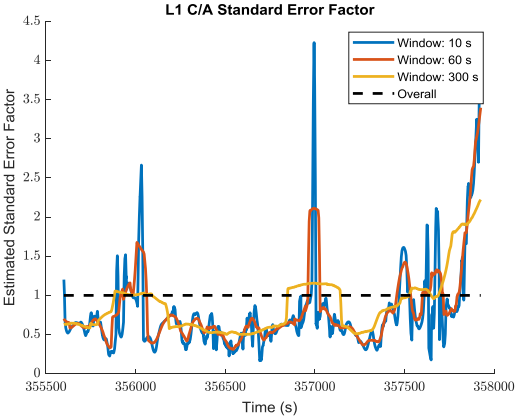

Figure 4.11: The time-varying standard error factor estimates for the L1 C/A observations.

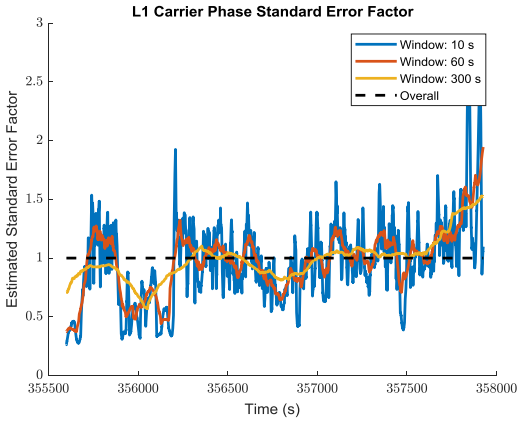

Figure 4.12: The time-varying standard error factor estimates for the L1 Carrier Phase observations.

Moreover, the separate standard error factors may be estimated for each of the three gyroscopes and accelerometers as long as they are uncorrelated with one another (Figure and Figure, respectively).

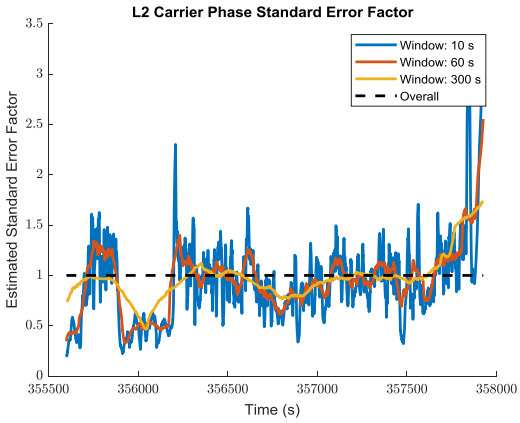

Figure 4.13: The time-varying standard error factor estimates for the L2 Carrier Phase observations.

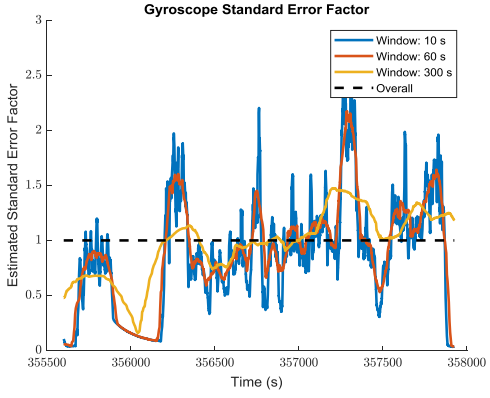

Figure 4.14: The time-varying standard error factor estimates for the Gyroscope observations.

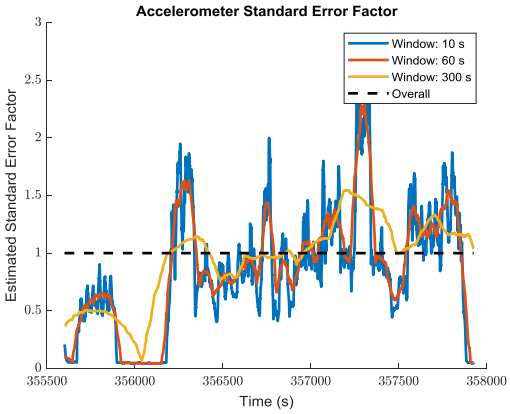

Figure 4.15: Time-varying standard error factor estimates for the Accelerometer observations.

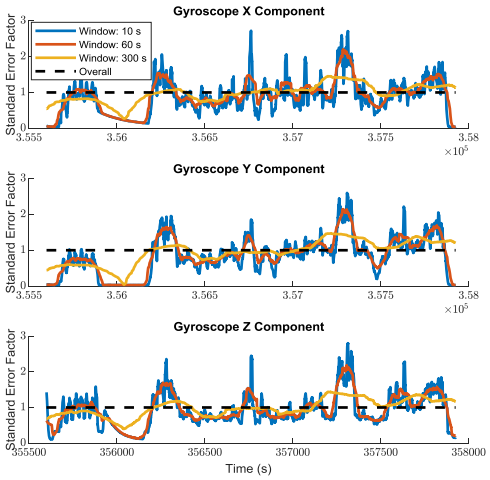

Figure 4.16: The time-varying standard error factor estimates for the individual components of the Gyroscope observations.

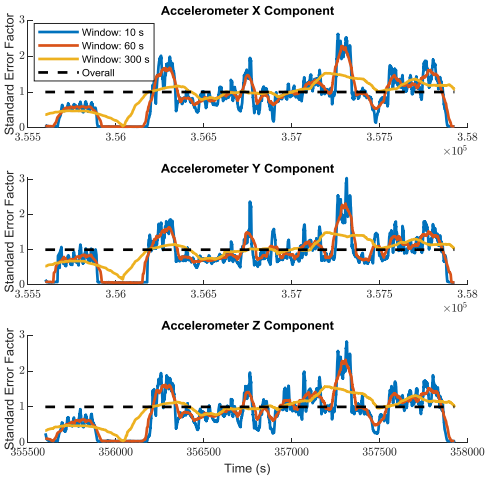

Figure 4.17: The time-varying standard error factor estimates for the individual components of the Accelerometer observations.

The estimated standard error factors for the process noise jerk and angular second derivative components are shown in Figure 4.18 and Figure 4.19, respectively.

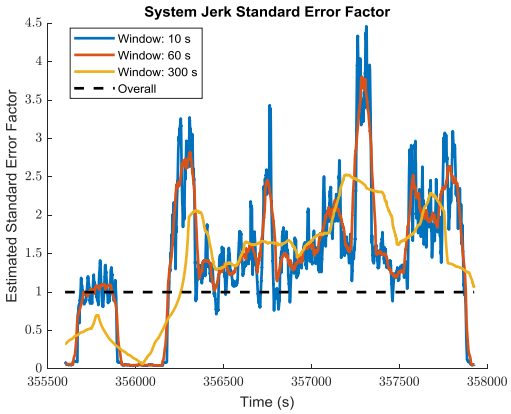

Figure 4.18: The time-varying standard error factor estimates for the process noise elements relating to the system jerk

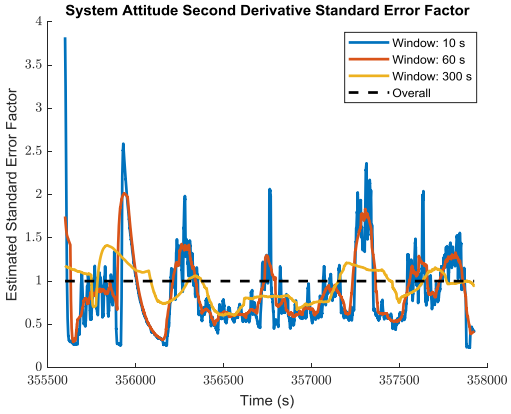

Figure 4.19: The time-varying standard error factor estimates for the process noise elements relating to the system attitude second derivatives

To illustrate the effects of applying the overall standard error factor to the estimated state covariance matrix, see the *a priori* and *a posteriori* standard deviation plots for the position (Figure 4.20) and attitude (Figure 4.21).

4.5 On Variance Component Convergence for Different *A Priori* Standard Errors

As long as the *a priori* estimates of the observation/process noise standard errors are relatively close to their *a posteriori* estimates, they should generally converge to the same values. To illustrate this, the same road test dataset was processed while over-estimating the observation standard errors. The *a priori* observation standard errors used are summarized in Table 4.5.

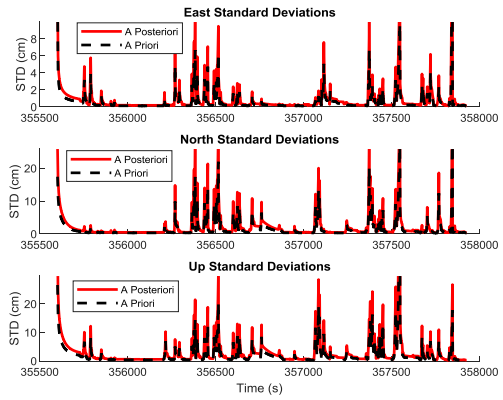

Figure 4.20: The time-varying standard deviations of the East, North, Up local geodetic coordinates of the system

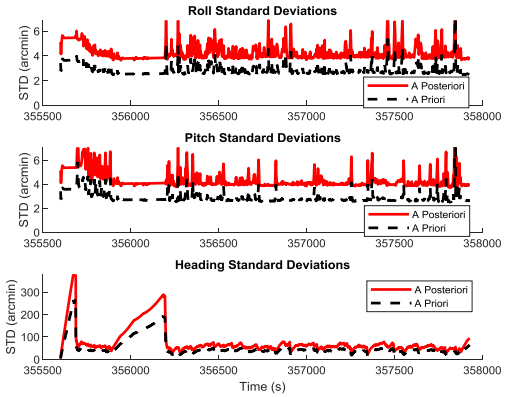

Figure 4.21: The time-varying standard deviations of the roll, pitch, and heading attitude parameters.

After 5 iterations of tuning the standard errors using variance component estimation, the estimated *a posteriori* observation standard errors are summarized in Table 4.6, along with their percentage difference from the values estimated in Table 4.3.

Table 4.5: The a priori observation standard errors used in data processing.

Observation Type	*A Priori* Standard Error
L1 C/A	±100 cm
L1 Carrier Phase	±15 mm
L2 Carrier Phase	±15 mm
Accelerometer	±15 cm/s^2
Gyroscope	±60’/s

Table 4.6 clearly illustrates that the *a posteriori* standard errors of the observations are primarily influenced by how well the collected data fits the observation models, and are uninfluenced by their initial *a priori* estimates as long as they have been well approximated based on user’s best knowledge of the being system and field conditions. This behaviour makes VCE very well-suited to adaptive filtering techniques.

Table 4.6: The a posteriori standard errors estimated for the initial values summarized in Table 4.5, and a comparison to the a posteriori standard errors described in Table 4.3.

Observation Type	*A Posteriori* Standard Error	Difference from Tab. 4.3 [%]
L1 C/A	±76 cm	2.7
L1 Carrier Phase	±11 mm	9.1
L2 Carrier Phase	±11 mm	0.0
Accelerometer	±12 cm/s^2	9.1
Gyroscope	±44 ‘/s	2.2

5. Conclusions and Remarks

This paper attempts to standardize the *a posteriori* precision evaluation process in discrete Kalman Filtering based on the standard Kalman filter described in Section 2, drawing from similar tasks in Least-Squares analysis. This enables the following a posteriori precision analysis:

1). Estimation of the redundancy of a Kalman Filter, as well as the redundancy contributions of the process noise, observation, and predicted state vectors.

2). Estimation of variance factors for each of the process noise, observation, and predicted state vectors, along with any independent partitioned elements of these vectors.

3). Evaluation of global variance factors that describe the overall performance of the Kalman Filter over an entire dataset.

4). Evaluation of local variance factors that describe the time-varying performance of the Kalman Filter over the course of a dataset. This includes windowed analysis to improve the reliability of variance factor estimates.

Each of them has been explored in the given working example based on a road test dataset.

It is worth noting that such a posteriori precision evaluation blends the a priori stochastic models and the quality of the measurements into a unity – as such, the delivered precision and accuracy of the solution in discrete Kalman filtering may well be related with the being processed data. This process does not necessarily

"improve" the precision and/or accuracy of the solution. Rather, our understanding of the Kalman Filtering system improves as a result of this proposed a posteriori precision evaluation, and this in turn improves the reliability of our results.

Acknowledgment: This research was partially supported by NSERC Discovery Grants Program – Individual (NSERC: the Natural Sciences and Engineering Research Council of Canada).

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