

Simplified Algorithms of Variance Component Estimation for Static and Kinematic GPS Single Point Positioning

Jian-Guo Wang¹ Nilesh Gopaul¹ Bruno Scherzinger²

¹ *Department of Earth and Space Science and Engineering, York University, Canada*

² *Applanix Corp., Canada*

Abstract

This paper adapts Helmert's simplified variance component estimation (VCE) algorithm for static and kinematic GPS single point positioning (SPP). First, the VCE algorithm for a static GPS SPP is formulated. Second, the concept of redundancy contribution of observations is developed in Kalman filtering so that the VCE algorithm is further delivered in Kalman filtering. The proposed VCE approach in Kalman filtering allows the variance components for individual measurement noises and individual independent process noises to be estimated. Some VCE numerical results from static and kinematic GPS datasets are presented and discussed.

Key words: variance component estimation, Kalman filter, redundancy contribution, GPS, static/kinematic point positioning.

1. Introduction

The algorithms for variance and covariance estimation (VCE) have attracted considerable research attentions over the 30 years. Examples include Helmert, 1907; Förstner, 1979; Grafarend, et al, 1980; Li, 1983; Koch, 1986; Jeudy, 1988; Qu, 1989; Yu, 1996; Xu, et al, 2006, 2007; Teunissen and Amiri-Simkooei, 2008; Amiri-Simkooei, 2007; etc.. Recently, more and more application-based studies have been presented [Sieg and Hirsch, 2000; Wang & Rizos, 2002; Tiberius, 2003; Rietdorf, 2004; Tesmer, 2004; Zhou, et al, 2006; Amiri-Simkooei, 2007; Bähr, et al, 2007; Böckmann, 2008; Milbert, 2008; etc.]. The VCE in Kalman filtering has been also studied because of its peculiar model architecture [Sage and Husa, 1969; Mehra and Peschon, 1971; Tsang, et al, 1981]. [Wang, 1997, Caspary, Wang, 1998] proposed a practical VCE algorithm

for Kalman filtering that can be applied not only to the variance components of the independent measurements and also to the variance components of the independent process noises. An adaptive Kalman filter based on VCE was developed and applied to GPS kinematic positioning [Hu & Liu, 2002; Hu, et al, 2003].

Many different methods have been developed for VCE based on different estimation principles. Usually VCE is associated with the complicated theoretical derivation and the complexity of the calculations. This manuscript addresses a more practical aspect of VCE.

Among various simplified VCE algorithms, the one based on the measurement redundancy contribution, is probably the most commonly used practical approach [Förstner, 1979]. Theoretically, it is a simplification of the rigorous Helmert method. This approach was well developed in the least squares adjustment and found applications in geodetic and navigation data processing [Li, 1984; Wang, 1997; Caspary and Wang, 1998; Sieg and Hirsch, 2000; Rietdorf, 2004; Tesmer, 2004; Zhou, et al, 2006; Bähr, et al, 2007; Böckmann, 2008; Milbert, 2008; etc.]. Experience has shown that it performs as good as the rigorous Helmert method wherever enough redundant measurements are available, and does not involve significant amount of computation.

The present contribution focuses on using simplified VCE algorithm for the determination of stochastic models for static and kinematic GPS SPP problems. Section 2 starts with an overview of the rigorous VCE methods since Helmert, the simplified VCE algorithm based on the measurement redundancy contribution is then outlined, followed by a VCE approach for sequential least squares. Section 3 gives a similar VCE algorithm in Kalman filtering. The introduction of the concept of the measurement redundancy contribution to Kalman filtering ensures the realization of this practical VCE algorithm. The proposed VCE algorithm allows for analysis of the measurement residuals and the estimated residuals of

process noise other than the system innovations. As a result, the variance components can be grouped into the individual measurements and the components in the process noise vector. Section 4 and Section 5 present VCE results from static and kinematic GPS SPP data and necessary discussions.

2. Helmert variance component estimation

2.1 Rigorous Algorithm after Helmert

The analysis starts with a generic linearized system of observation equations

$$\mathbf{L} + \mathbf{v} = \mathbf{B}d\hat{\mathbf{x}} + \mathbf{F}(\mathbf{x}^{(0)}) \quad (1)$$

where \mathbf{L} is the $n \times 1$ observation vector; \mathbf{v} is the $n \times 1$ residual vector of \mathbf{L} ; \mathbf{x} is the $t \times 1$ parameter vector with a vector $\mathbf{x}^{(0)}$ of known approximate values and the correction vector $d\hat{\mathbf{x}}$ for $\mathbf{x}^{(0)}$; $\mathbf{F}(\mathbf{x})$ is the $n \times 1$ vector as nonlinear mathematical function of \mathbf{x} for \mathbf{L} ; \mathbf{B} is the $n \times t$ design matrix that is composed of the partial derivatives of $\mathbf{F}(\mathbf{x})$ with respect to \mathbf{x} at $\mathbf{x}^{(0)}$. The observation vector \mathbf{L} is normally distributed as $\mathbf{L} \sim N(\tilde{\mathbf{L}}, \mathbf{D}_{LL})$ with its expectation vector $\tilde{\mathbf{L}}$ and its variance matrix \mathbf{D}_{LL} . In practice, \mathbf{D}_{LL} is given as

$$\mathbf{D}_{LL} = \mathbf{S}_0^2 \mathbf{P}^{-1} \quad (2)$$

where \mathbf{S}_0^2 is the variance of unit weight and \mathbf{P} is the weight matrix of \mathbf{L} . The inverse of a weight matrix is called as a cofactor matrix and interchangeably used together with the weight matrix.

The least-squares (LS) solution of (1) is

$$d\hat{\mathbf{x}} = \mathbf{N}^{-1} \mathbf{B}^T \mathbf{P} \mathbf{L} \quad (3)$$

with its variance matrix

$$\mathbf{D}_{\hat{\mathbf{x}}} = \hat{\mathbf{S}}_0^2 \mathbf{N}^{-1} \quad (4)$$

where

$$\mathbf{N} = \mathbf{B}^T \mathbf{P} \mathbf{B} \quad (5)$$

$$\hat{\mathbf{S}}_0^2 = \frac{\mathbf{v}^T \mathbf{P} \mathbf{v}}{n - t} \quad (n > t) \quad (6)$$

$$\mathbf{l} = \mathbf{L} - \mathbf{F}(\mathbf{x}^{(0)}) \quad (7)$$

The model from (1) to (7) only deals with the variance of unit weight \mathbf{S}_0^2 because \mathbf{P} is assumed known and most likely diagonal that represents \mathbf{L} being composed of n independent measurements.

The a priori measurement weight or variance matrix is determined on the basis of the available stochastic information about the measurements. In general, the available stochastic information is limited and needs to be improved. Fortunately, the variances for individual measurements or the variance components for the grouped measurements can posteriorly be estimated based on the measurement residuals in addition to achieving the LS solution of the parameters.

The Helmert method is probably the most popular VCE algorithm [Förstner, 1979; Koch, 1986; Cui et al, 2001; Båhe, et al, 2007; etc.]. Consider m types of measurements as the $n_i \times 1$ vector \mathbf{L}_i for $i = 1, 2, \dots, m$. The equation system (1) can be partitioned as follows

$$\begin{pmatrix} \mathbf{L}_1 \\ \vdots \\ \mathbf{L}_i \\ \vdots \\ \mathbf{L}_m \end{pmatrix} + \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1 \\ \vdots \\ \mathbf{B}_i \\ \vdots \\ \mathbf{B}_m \end{pmatrix} d\hat{\mathbf{x}} + \begin{pmatrix} \mathbf{F}_1(\mathbf{x}^{(0)}) \\ \vdots \\ \mathbf{F}_i(\mathbf{x}^{(0)}) \\ \vdots \\ \mathbf{F}_m(\mathbf{x}^{(0)}) \end{pmatrix} \quad (8)$$

with (2) partitioned in

$$\mathbf{D}_{LL} = \text{diag}(\mathbf{S}_{01}^2 \mathbf{P}_1^{-1} \quad \dots \quad \mathbf{S}_{0i}^2 \mathbf{P}_i^{-1} \quad \dots \quad \mathbf{S}_{0m}^2 \mathbf{P}_m^{-1}) \quad (9)$$

(9) indicates that $\mathbf{L}_1, \dots, \mathbf{L}_i, \dots, \mathbf{L}_m$ do not share the same variance of unit weight \mathbf{S}_0^2 . The purpose of the variance component estimation is to estimate the variance factors \mathbf{S}_{0i}^2 ($i = 1, 2, \dots, m$) and iteratively or adaptively adjust the measurement weights or variances using the available measurements. They can be estimated from the following equation [Förstner, 1979; Cui, et al, 2001; Båhr, et al, 2007]:

$$\begin{pmatrix} \mathbf{v}_1^T \mathbf{P}_1 \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m^T \mathbf{P}_m \mathbf{v}_m \end{pmatrix} = \begin{pmatrix} \mathbf{S}_{11} & \dots & \mathbf{S}_{1i} & \dots & \mathbf{S}_{1m} \\ \vdots & & \vdots & & \vdots \\ \mathbf{S}_{i1} & \dots & \mathbf{S}_{ii} & \dots & \mathbf{S}_{im} \\ \vdots & & \vdots & & \vdots \\ \mathbf{S}_{m1} & \dots & \mathbf{S}_{mi} & \dots & \mathbf{S}_{mm} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{S}}_{01}^2 \\ \vdots \\ \hat{\mathbf{S}}_{0i}^2 \\ \vdots \\ \hat{\mathbf{S}}_{0m}^2 \end{pmatrix} \quad (10)$$

where

$$\mathbf{S}_{ii} = n_i - 2\text{tr}(\mathbf{N}^{-1} \mathbf{N}_i) + \text{tr}(\mathbf{N}^{-1} \mathbf{N}_i \mathbf{N}_i^{-1} \mathbf{N}_i) \quad (11)$$

$$\mathbf{S}_{ij} = \mathbf{S}_{ji} = \text{tr}(\mathbf{N}^{-1} \mathbf{N}_i \mathbf{N}_i^{-1} \mathbf{N}_j) \quad (i \neq j) \quad (12)$$

$$\mathbf{N}_i = \mathbf{B}_i^T \mathbf{P}_i \mathbf{B}_i \quad (13)$$

for $i, j = 1, \dots, m$.

2.2. The simplified algorithm

The inverse of the $m \times m$ coefficient matrix in (10) makes the method inconvenient. Correspondingly, a number of simplified formulas have been developed [Förstner, 1979; Cui et al, 2001; Bähr, et al, 2007]. The most popular simplification was developed on the basis of the measurement redundant contribution by Förstner [1979]. For more details about Förstner's algorithm along with its important characteristics refer to [Förstner, 1979; Cui et al, 2001; Bähr, et al, 2007; etc.]. For the needs of further development in this manuscript, a brief overview is given in the following.

By applying the law of the error propagation to the least squares solution of (8) and (9), the variance matrix of the measurement residual vector \mathbf{v}_i can be obtained

$$\mathbf{D}_{\mathbf{v}_i \mathbf{v}_i} = \mathbf{S}_{0i}^2 \mathbf{Q}_{\mathbf{v}_i \mathbf{v}_i} = \mathbf{S}_{0i}^2 (\mathbf{P}_i^{-1} - \mathbf{B}_i \mathbf{N}^{-1} \mathbf{B}_i^T) \quad (14)$$

wherein $\mathbf{Q}_{\mathbf{v}_i \mathbf{v}_i}$ is the cofactor matrix of \mathbf{v}_i . The expected value of the weighted sum of squared residuals from the group i is

$$E(\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i) = \mathbf{S}_{0i}^2 [n_i - \text{trace}(\mathbf{N}^{-1} \mathbf{N}_i)] \quad (15)$$

where n_i is the number of the measurements in the group i . It can be proved that the following equation is satisfied

$$n_i - \text{trace}(\mathbf{N}^{-1} \mathbf{N}_i) = \text{trace}(\mathbf{P}_i \mathbf{Q}_{\mathbf{v}_i \mathbf{v}_i}) = r_i \quad (16)$$

Herewith r_i is equal to the redundancy contribution made by \mathbf{L}_i to the system (1) and called as the redundant index of \mathbf{L}_i . Hence, the estimated variance factor \mathbf{S}_{0i}^2 is

$$\mathbf{S}_{0i}^2 = \frac{\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{r_i} \quad (i = 1, \dots, m) \quad (17)$$

Obviously, (17) is easy to be applied because it does not introduce extra calculations since the redundant contribution of measurements is always required in data processing for the purpose of outlier detection or reliability analysis. In general, all of the VCE algorithms need enough redundant measurements to provide reliable results, so does (17).

Based on the fact that there are usually a large number of redundant measurements in GPS positioning, the authors confidently propose adopting Förstner's algorithm for VCE in GPS SPP. Accordingly, the practical algorithms are developed for static GPS SPP in Section 2.3 and for kinematic GPS SPP in Section 3.

2.3. The VCE in Sequential Least Squares Method

The focus now is turned to construct the simplified algorithm in sequential LS. Given an equation system for n groups of measurements as

$$\begin{bmatrix} \mathbf{a} \mathbf{v}_1 \mathbf{0} \\ \vdots \\ \mathbf{c} \mathbf{v}_k \mathbf{0} \\ \vdots \\ \mathbf{c} \mathbf{v}_n \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{a} \mathbf{B}_{10} & \mathbf{B}_{11} & \cdots & \mathbf{O} & \cdots & \mathbf{O} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{c} \mathbf{B}_{k0} & \mathbf{O} & \cdots & \mathbf{B}_{kk} & \cdots & \mathbf{O} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{c} \mathbf{B}_{n0} & \mathbf{O} & \cdots & \mathbf{O} & \cdots & \mathbf{B}_{nn} \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_k \\ \vdots \\ \mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \mathbf{l}_1 \\ \vdots \\ \mathbf{l}_k \\ \vdots \\ \mathbf{l}_n \end{bmatrix} \quad (18)$$

with the variance matrix for all the measurements together is assumed as

$$\mathbf{D} = \text{diag}\{\mathbf{D}_{11} \quad \cdots \quad \mathbf{D}_{kk} \quad \cdots \quad \mathbf{D}_{nn}\} \\ = \text{diag}\{\mathbf{S}_{01}^2 \mathbf{P}_1^{-1} \quad \cdots \quad \mathbf{S}_{0k}^2 \mathbf{P}_k^{-1} \quad \cdots \quad \mathbf{S}_{0n}^2 \mathbf{P}_n^{-1}\} \quad (19)$$

in which there are n -group measurements and the sub-vector \mathbf{x}_0 is the common parameters in the individual groups, \mathbf{v}_k is the measurement residual vector in the group k , \mathbf{x}_k is the parameter vector owned only by the group k , and \mathbf{B}_{kk} is the coefficient matrix.

Equation (18) is a typical structure for the multiple epochs of static GPS SPP where \mathbf{x}_0 contains the three receiver coordinates and \mathbf{x}_k is the receiver clock error at epoch k ($k = 1, 2, \dots, n$). At an arbitrary epoch k , the measurement variance matrix is a diagonal positive matrix

$$\mathbf{D}_{kk} = \text{diag}\{\mathbf{S}_{k1}^2 \quad \cdots \quad \mathbf{S}_{ki}^2 \quad \cdots \quad \mathbf{S}_{kn_k}^2\} \quad (20)$$

Accordingly, the variance factors can be grouped, for example, to associate the pseudorange measurements with the same satellite in a group within these n epochs in (19).

Rather than for (18) as a whole, one usually runs a sequential solution. Without giving the detailed algorithm derivation, it here relies upon the VCE for the individual groups in a sequential LS fashion analogous to (17). The accumulative variance factor is estimated as:

$$\hat{\mathbf{S}}_i^2 = \frac{\sum_{k=1}^n \mathbf{v}_{ik}^2}{n} \quad (i = 1, 2, \dots) \quad (21)$$

where \mathbf{v}_{ik} and r_{ik} denote the residual and redundant index of the i -th measurement at epoch k , respectively. Basically, the estimated \mathbf{x}_0 from the past epoch will be integrated with the measurements at the current epoch as pseudo-measurements.

3. THE VCE IN KALMAN FILTERING

The simplified algorithm in Section 2.2 is now introduced for the VCE in Kalman filtering since it is expected that a large number of measurement epochs could be available to provide enough redundant measurements. The key is still to compute the redundancy contribution and the residuals of the measurements. These are not usual quantities explicitly given in Kalman filtering, but are referred to [Wang, 1997; Caspary, Wang, 1998].

3.1. The Kalman Filter

A linear or linearized system with state-space notation is considered with the data available over a discrete time series $\{t_0, t_1, \dots, t_N\}$, which will often be simplified to $\{0, 1, \dots, N\}$. Without loss of generality, a deterministic system input vector will be removed from the expression. Hence, at any time instant k ($1 \leq k \leq N$) the system can be written as follows:

$$\mathbf{x}(k) = \mathbf{A}(k-1)\mathbf{x}(k-1) + \mathbf{B}(k-1)\mathbf{w}(k-1) \quad (22)$$

$$\mathbf{z}(k) = \mathbf{C}(k)\mathbf{x}(k) + \Delta(k) \quad (23)$$

where $\mathbf{x}(k)$ is the n state-vector; $\mathbf{z}(k)$ is the p observation vector, $\mathbf{w}(k-1)$ is the m -process noise vector, $\Delta(k)$ is the p measurement noise vector; $\mathbf{A}(k-1)$ is the $n \times n$ coefficient matrix of $\mathbf{x}(k)$; $\mathbf{B}(k-1)$ is the $n \times m$ coefficient matrix of $\mathbf{w}(k)$; $\mathbf{C}(k)$ is the $p \times n$ coefficient matrix of $\mathbf{x}(k)$ in the observation equation. The random vectors $\mathbf{w}(k-1)$ and $\Delta(k)$ are generally assumed to be Gaussian with zero-mean: $\mathbf{w}(k-1) \sim N(\mathbf{o}, \mathbf{Q}(k-1))$ and $\Delta(k) \sim N(\mathbf{o}, \mathbf{R}(k))$ with $\mathbf{Q}(k-1)$ and $\mathbf{R}(k)$ positive definite, respectively. Further assumptions about the random noise are specified as follows: $\text{Cov}(\mathbf{w}(i), \mathbf{w}(j)) = \mathbf{O}$, $\text{Cov}(\Delta(i), \Delta(j)) = \mathbf{O}$ and $\text{Cov}(\mathbf{w}(i), \Delta(j)) = \mathbf{O}$ for $i \neq j$. Commonly, one also assumes to have the initial mean and variance-covariance matrix $\mathbf{x}(0)$ and $\mathbf{D}_{xx}(0)$ for the system state at the time epoch 0 available and to have $\mathbf{x}(0)$ independent of $\mathbf{w}(k-1)$ and $\Delta(k)$ at any epoch k .

Under the assumptions given above, the optimal estimate $\hat{\mathbf{x}}(k)$ of $\mathbf{x}(k)$ can straightforward be derived in the sense of unbiasedness and minimum variance. However, there is no necessity to provide any further detail here.

3.2. An Alternate Prospect of Kalman Filter

Let us analyze the stochastic information in Kalman filtering in a different way. The optimal estimate $\hat{\mathbf{x}}(k)$ of $\mathbf{x}(k)$ at the instant k is always associated with the

stochastic information that may be divided into three independent groups:

- The observation noise $\Delta(k)$,
- The system noise $\mathbf{w}(k-1)$,
- The noise on the predicted $\hat{\mathbf{x}}(k/k-1)$ through $\hat{\mathbf{x}}(k-1)$, on which $\{\Delta(1), \Delta(2), \dots, \Delta(k-1)\}$ and $\{\mathbf{w}(0), \mathbf{w}(1), \dots, \mathbf{w}(k-2)\}$ are propagated into the current state vector.

Traditionally, “b” and “c” will be considered together in $\hat{\mathbf{x}}(k/k-1)$, where $(k/k-1)$ means the one step prediction from time $k-1$ to k . In fact, these three different error resources can be studied separately. Along with this line of thinking, the system model with (22) and (23) can be reformulated through three groups of the observation residual equations as follows [Wang, 1997; Caspary & Wang, 1998; Wang, 2008]:

$$\left. \begin{aligned} \mathbf{v}_{l_x}(k) &= \hat{\mathbf{x}}(k) - \mathbf{B}(k-1)\hat{\mathbf{w}}(k-1) - \mathbf{I}_x(k) \\ \mathbf{v}_{l_w}(k) &= \hat{\mathbf{w}}(k-1) - \mathbf{I}_w(k) \\ \mathbf{v}_{l_z}(k) &= \mathbf{C}(k)\hat{\mathbf{x}}(k) - \mathbf{I}_z(k) \end{aligned} \right\} \quad (24)$$

where the independent (pseudo-)observation groups are defined by

$$\left. \begin{aligned} \mathbf{I}_x(k) &= \mathbf{A}(k-1)\hat{\mathbf{x}}(k-1) \\ \mathbf{I}_w(k) &= \mathbf{w}_0(k-1) \\ \mathbf{I}_z(k) &= \mathbf{z}(k) \end{aligned} \right\} \quad (25)$$

with their variance-covariance matrices

$$\left. \begin{aligned} \mathbf{D}_{l_x l_x}(k) &= \mathbf{A}(k-1)\mathbf{D}_{xx}(k-1)\mathbf{A}^T(k-1) \\ \mathbf{D}_{l_w l_w}(k) &= \mathbf{Q}(k-1) \\ \mathbf{D}_{l_z l_z}(k) &= \mathbf{R}(k) \end{aligned} \right\} \quad (26)$$

where $\mathbf{D}_{xx}(k-1)$ is the variance matrix of the estimated state vector $\hat{\mathbf{x}}(k-1)$ at time $k-1$. $\mathbf{I}_x(k)$, $\mathbf{I}_w(k)$ and $\mathbf{I}_z(k)$ are the n -, m - and p -dimensional measurement or pseudo-measurement vectors, respectively. Usually, one has $\mathbf{w}_0(k) = \mathbf{o}$.

By applying the principle of least squares to (24)~(26), the identical $\hat{\mathbf{x}}(k)$ can be obtained for $\mathbf{x}(k)$ as in Kalman filtering [Wang, 1997, 2008; Caspary & Wang, 1998].

This alternate formulation directly makes the measurement residual vectors available for error analysis. The measurement residual vectors are the following functions of the system innovation vector at each epoch

$$\mathbf{v}_{l_x l_x}(k) = \mathbf{D}_{l_x l_x}(k)\mathbf{D}_{xx}^{-1}(k/k-1)\mathbf{K}(k)\mathbf{d}(k) \quad (27)$$

$$v_{l_w l_w}(k) = Q(k-1)B^T(k-1)D_{xx}^{-1}(k/k-1)K(k)d(k) \quad (28)$$

$$v_{l_z l_z}(k) = \{C(k)K(k) - E\}d(k) \quad (29)$$

where

$$D_{xx}(k/k-1) = A(k-1)D_{xx}(k-1)A^T(k-1) + B(k-1)Q(k-1)B^T(k-1) \quad (30)$$

$$K(k) = D_{xx}(k/k-1)C^T(k)D_{dd}^{-1}(k) \quad (31)$$

$$d(k) = z(k) - A(k-1)\hat{x}(k-1) \quad (32)$$

$$D_{dd}(k) = C(k)D_{xx}(k/k-1)C^T(k) + R(k) \quad (33)$$

The variance matrices for (27), (28) and (29) are

$$D_{v_{l_x l_x}}(k) = A(k-1)D_{xx}(k-1)A^T(k-1)C^T(k) \quad (34)$$

$$D_{dd}^{-1}(k)C(k)A(k-1)D_{xx}(k)A^T(k-1) \quad (32)$$

$$D_{v_{l_w l_w}}(k) = Q(k-1)B^T(k-1)C^T(k)D_{dd}^{-1}(k) \quad (35)$$

$$C(k)B(k-1)Q(k-1) \quad (33)$$

$$D_{v_{l_z l_z}}(k) = \{E - C(k)K(k)\}R(k) \quad (34)$$

In Kalman filter, $d(k)$ and $K(k)$ are called as the system innovation vector and the gain matrix, respectively. The measurement residuals are characterized as an uncorrelated series epochwise

$$Cov\{v(i), v(j)\} = O(i-1, j) \quad (35)$$

3.3. The Redundancy Contribution in Kalman Filtering

The redundant index for each of the measurement components in $z(k)$ and $w(k-1)$ is given by

$$r_{z_i}(k) = 1 - \{C(k)K(k)\}_{ii} \quad (36)$$

$$r_{w_i}(k) = [Q(k-1)B^T(k-1)C^T(k)D_{dd}^{-1}(k)C(k)B(k-1)]_{ii} \quad (37)$$

under the assumption that $Q(k-1)$ and $R(k)$ are diagonal. The total redundancy contribution of $l_x(k)$ cannot be decomposed to its individual redundant indices because their components are correlated in general.

For three independent measurement groups as in (25), the individual group redundant indices are:

$$r_x(k) = trace[A(k-1)D_{xx}(k)A^T(k-1)C^T(k)D_{dd}^{-1}(k)C(k)] \quad (38)$$

$$r_w(k) = trace[Q(k)B^T(k)C^T(k)D_{dd}^{-1}(k)C(k)B(k-1)] \quad (39)$$

$$r_z(k) = trace[I - C(k)K(k)] \quad (40)$$

One can easily prove:

$$r(k) = r_x(k) + r_w(k) + r_z(k) = p(k) \quad (41)$$

where $p(k)$ is the number of the total redundant measurements at epoch k and equal to the dimension of the real measurement vector $z(k)$.

3.4. The Variance Component Estimation

For an arbitrary epoch k , the individual variance factors for $z(k)$ can be estimated by

$$\hat{S}_{z_i z_i}^2(k) = \frac{v_{z_i z_i}^2(k)}{r_{z_i}(k)} \quad (i = 1, \dots, p) \quad (42)$$

The accumulative individual variance factor in $z(k)$ can be estimated from the past k epochs:

$$\hat{S}_{z_i z_i}^2(1, \dots, k) = \frac{v_{z_i z_i}^2(1) + \dots + v_{z_i z_i}^2(k)}{r_{z_i}(1) + \dots + r_{z_i}(k)} \quad (i = 1, \dots, p) \quad (43)$$

For $w(k-1)$, the similar formulas can be given by:

$$\hat{S}_{w_i w_i}^2(k) = \frac{v_{w_i w_i}^2(k)}{r_{w_i}(k)} \quad (i = 1, \dots, m) \quad (45)$$

and

$$\hat{S}_{w_i w_i}^2(1, \dots, k) = \frac{v_{w_i w_i}^2(1) + \dots + v_{w_i w_i}^2(k)}{r_{w_i}(1) + \dots + r_{w_i}(k)} \quad (i = 1, \dots, m) \quad (46)$$

4. VCE in static GPS SPP

The VCE approach proposed in this paper is suitable for situations where the individual measurements are uncorrelated. In GPS SPP this is an appropriate assumption for the raw measurements and combinations of two different types of measurements, for example the ionosphere free combination of L1 C/A code and L1 carrier phase.

A software utility has been implemented for the algorithms described in Section 2 for GPS SPP processing. A 2-hour dataset from a receiver collected on July 1, 2008 is used to show the numerical performance. The ionosphere free combination using L1 C/A and L1 carrier phases was used as the measurements. The precise orbits and clocks were employed. The slant troposphere delay was corrected with a Saastamonien model for zenith troposphere delay and a Neil mapping function.

An integrated solution for the whole dataset was obtained by applying the least squares method as in Figure 1.

The initial variance matrix for all of the measurements was assigned to be identity that is equivalent to a measurement accuracy of 1m. The variance components were grouped against the satellite elevation angles of every 5-degree and 10-degree intervals, respectively. The upper plot in Fig. 1 shows the VCE results from the rigorous Helmert algorithm tagged as “HM” and from the simplified algorithm tagged as “SM”. The simplified

algorithm VCE fits closely to that from the rigorous Helmert algorithm. The lower plot in Fig. 1 gives the number of the total available measurements corresponding to the individual elevation intervals. For further analysis on these results, an exponential curve can be fitted to study how these variances are related to the elevation angles.

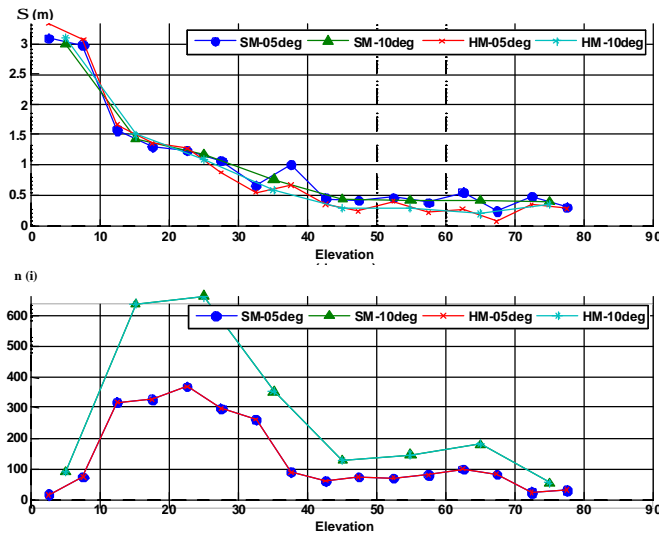


Fig. 1: VCE: LS Simplified vs. Helmut

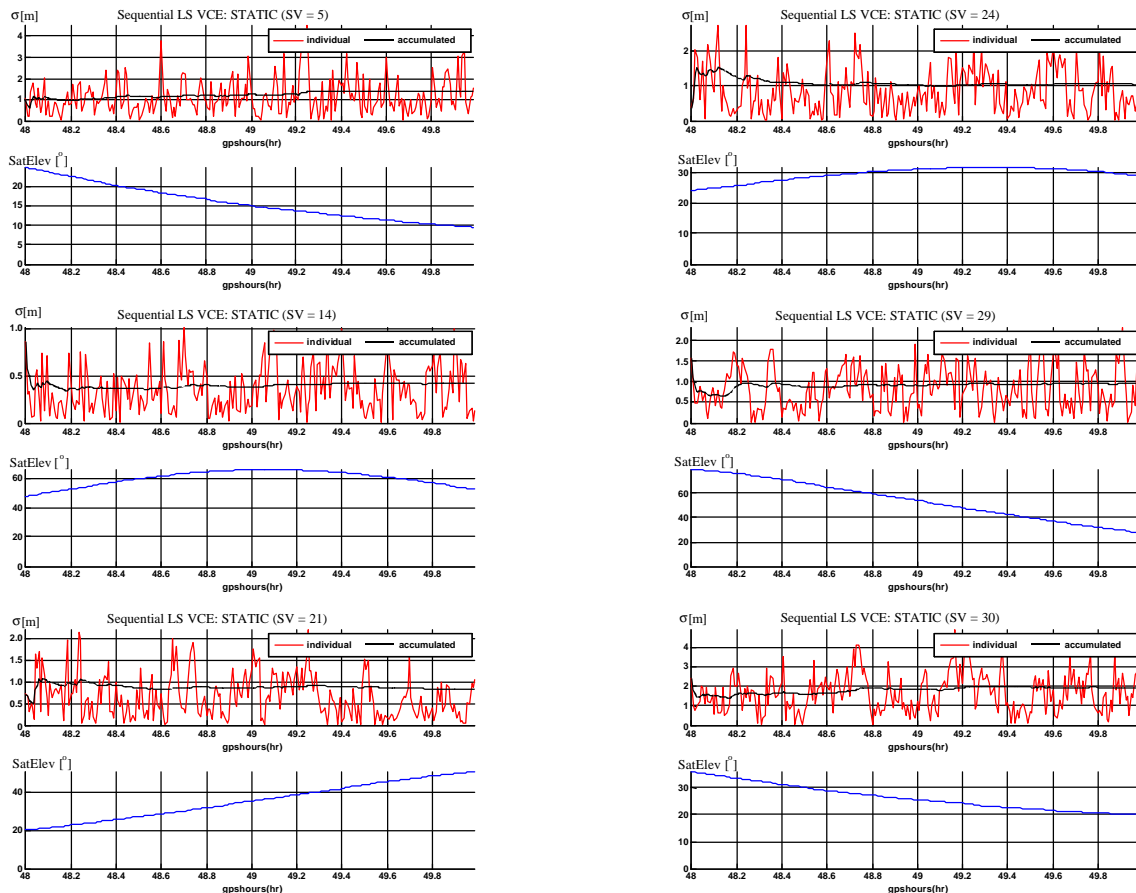


Fig. 2: The standard deviations from sequential LS VCE at the 5th run

The sequential algorithm from Section 3.2 was applied to the same dataset in terms of the variance components with respect to individual satellites. Here the initial variances were assigned to be identity for each of the measurements at each epoch. The estimated global variances were taken as the next initial variances to iteratively run the data processing five times for the purpose of the analysis. Fig. 2 gives a group of VCE results with individual accumulative standard deviations corresponding to the available satellites and their elevation angles from the 5th iteration whilst Fig. 3 shows the global posteriori VCE results for 6 of 12 available satellites from 5 iterations.

The purpose of this example is to show the convergence of the proposed VCE algorithm. As can be seen in Fig. 3, each of the estimated standard deviation factors has converged well. However, the results in Fig. 3 do not exhibit dependency on the elevation angles for long time periods as one might expect because the variance components were selected for each of the satellites without having considered the effects of their elevation changes. An alternate way is to set up the elevation interval in which the measurements are assumed to have the same accuracy and to perform VCE not only for each of the measurements over the entire time duration, but also taking different elevation angle intervals into account.

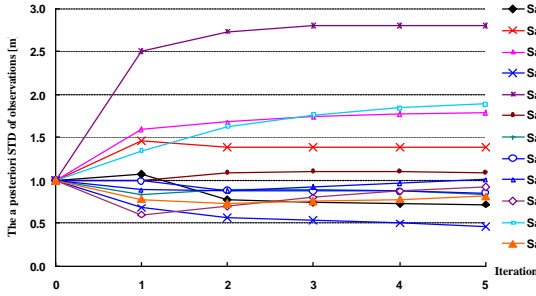


Fig. 3: The results from sequential LS VCE[m]

5. VCE in kinematic GPS SPP

A kinematic dataset (courtesy of Applanix Corp.) with the date rate 1Hz for 95 minutes on March 08, 2009 from Richmond Hill, Ontario, was analyzed using the VCE approach in Section 3. Similarly to the static dataset, the ionosphere free combination using L1 C/A and L1 carrier phases were used as the measurements. The troposphere delay was corrected with a Neil mapping function applied to a Saastamonien model for the zenith troposphere delay. The precise orbits and clocks were employed.

A position-velocity filter was used to estimate the receiver position, velocity, receiver clock bias and drift. Three acceleration components and the receiver clock drift change are modeled as the process noise vector. The variance components for each of the satellites and each of the process noise factors were estimated. All of the initial values were assigned to be identity that is equivalent to a priori accuracy of 1m, and process noise standard

deviations of 1m/s^2 for three accelerations and 1s/s^2 for the clock drift change. The estimated global variances were taken as the next initial variances to iteratively run the data processing five times. Fig. 4 shows the VCE results. The results for 13 available satellites have converged well (Fig. 4(a)), so have the ones for three acceleration components and the change of the clock drift for the receiver position as process noises (Fig. 4(b)).

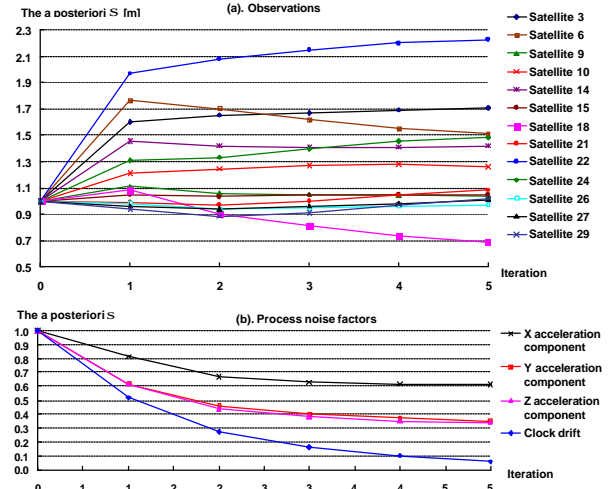


Fig. 4: The VCE results from kinematic SPP

However, the estimated variance for one of the process noise factors, the change of the clock drift, is relatively small because the used receiver does not exhibit a significant clock drift variation. This may result in the divergence of the estimated variance while its redundancy contribution becomes increasingly small if the variance factor becomes very small (the equivalent of a high leverage random variable as in linear regression). In order to avoid a potential divergence of this type of variance component, once can simply exclude it from the VCE process by fixing its value. Fig. 5 summarized the results with the fixed standard deviation ($S_{dr_i} = 0.095$) for the change of the clock drift. No significant differences can be found between Fig. 4 and Fig. 5.

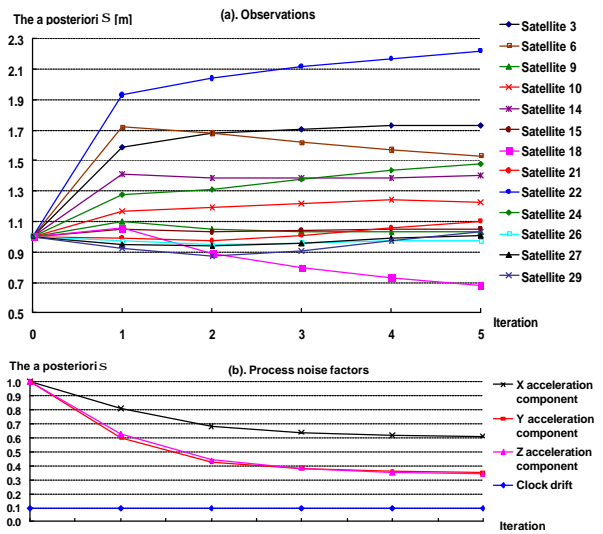


Fig. 5: The VCE results from kinematic SPP

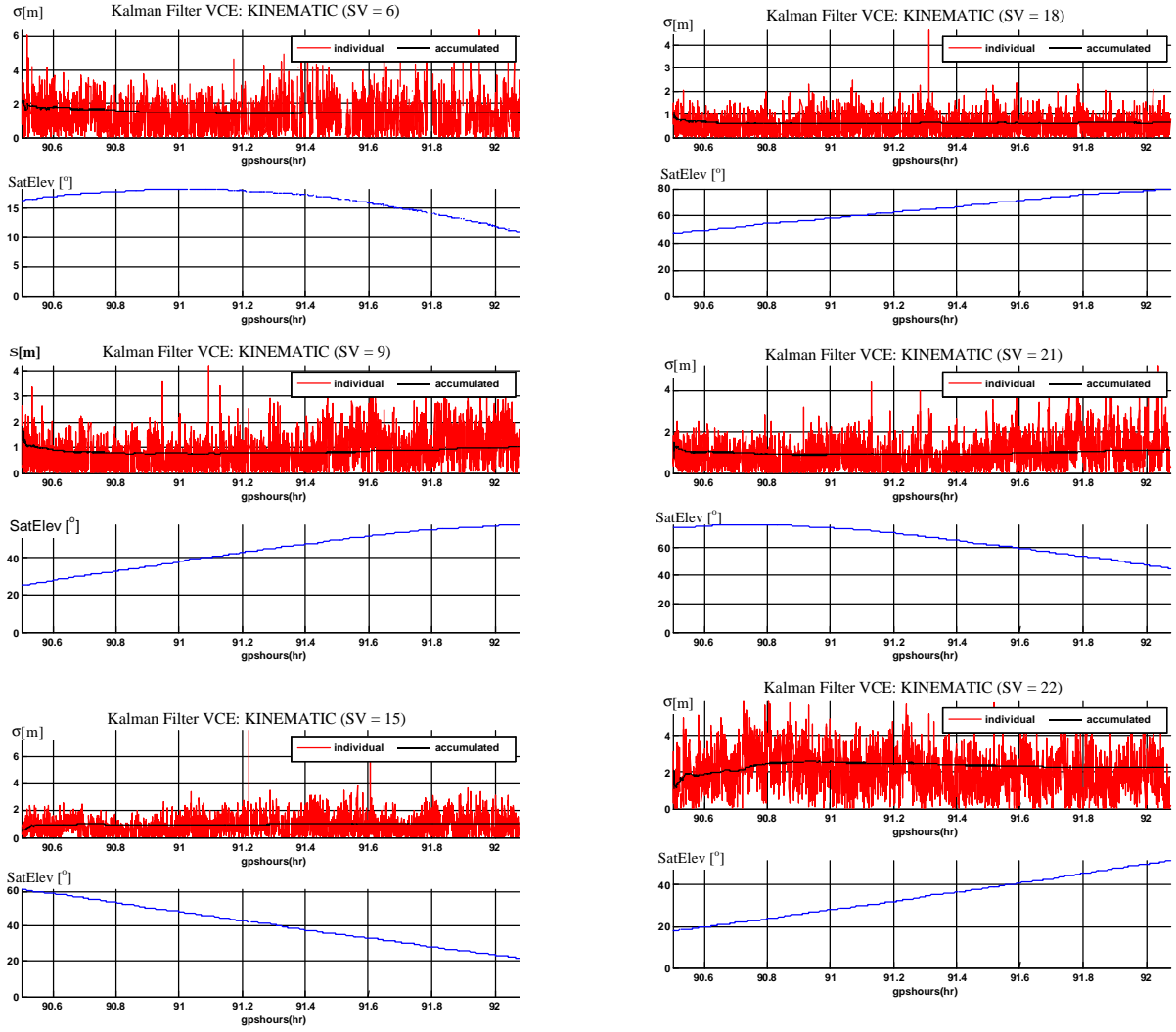
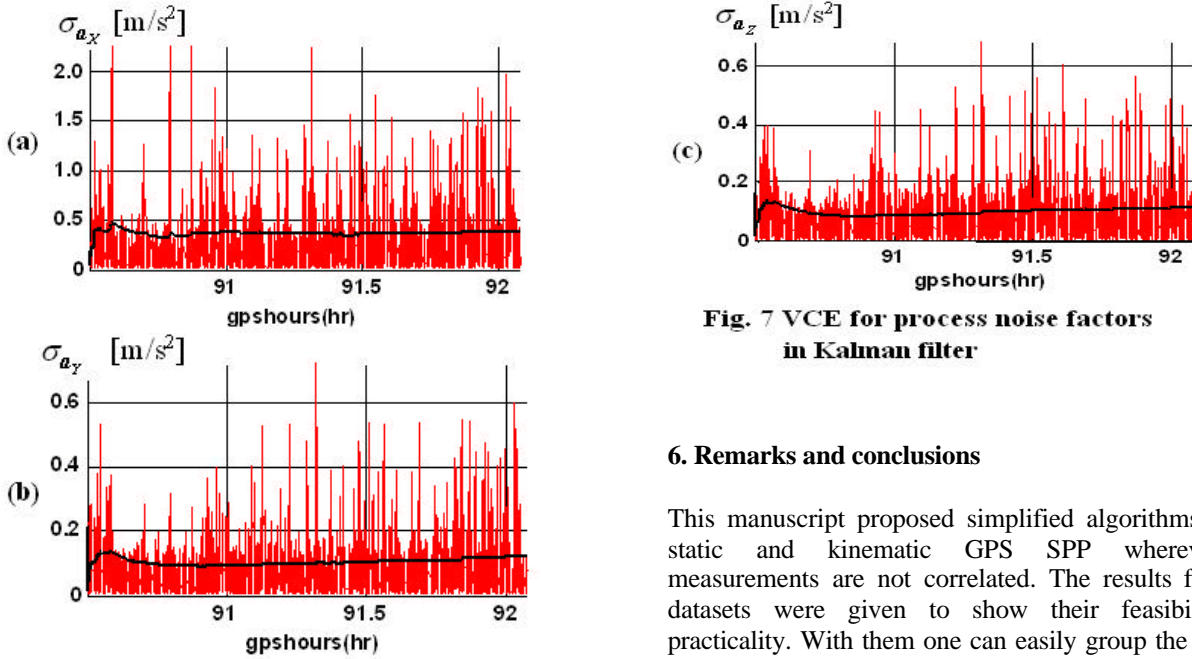
Fig. 6: The standard deviations from Kalman filter VCE at the 5th run

Fig. 7 VCE for process noise factors in Kalman filter

6. Remarks and conclusions

This manuscript proposed simplified algorithms for the static and kinematic GPS SPP wherever the measurements are not correlated. The results from real datasets were given to show their feasibility and practicality. With them one can easily group the variance

components based on the nature of the applications and user's objectives.

First, the proposed algorithm for the sequential least-squares method allows dealing with a low dimensioned parameter vector. Second, the algorithm formulated for VCE in Kalman filtering has its advantages that directly connect the error analysis with the measurement residuals other than the system innovation vector. This offers the capability to estimate not only the variance components for the measurement vector, but also the variance components for the process noise factors. Based on the fact that GPS data processing typically comprises a high volume of redundant measurements, the proposed algorithms are efficient without requiring a lot of computation and have good convergence properties. The posteriori variance components can be used either to evaluate the overall mission performance and/or create a practical adaptive process either in the sequential least-squares method or in a Kalman filter.

Future research will address the development of practical algorithms for the estimation of variance and covariance components that can be applied to single differenced PPP models and double differenced RTK GPS models, especially for kinematic applications using a Kalman filter.

7. Reference

- Amiri-Simkooei (2007), *Least-squares variance component estimation –Theory and GPS Applications*, PhD thesis, Delft University of Technology, 2007.
- Caspary, Wilhelm; Wang, Jianguo (1998), *Redundanzanteile und Varianzkomponenten im Kalman Filter*, Zeitschrift für Vermessungswesen (ZfV), Vol. 123, No.4, 1998, pp.121-128.
- Cui, Xi-Zhang, et al (2001), *Generalized Surveying Adjustment*, Textbook, the Publishing House of WTUSM, Wuhan, 2001 (in Chinese).
- Bähr, Hermann; Altamimi, Zuheir; Heck, Bernhard (2007), *Variance Component Estimation for Combination of Terrestrial Reference Frames*, Schriftenreihe des Studiengangs Geodäsie und Geoinformatik, Universität Karlsruhe, No. 6, 2007.
- Böckmann, Sarah (2008), *Die Varianzkomponenten-schätzung in der IVS-Kombination*, Workshop 2008 der Forschungsgruppe Satellitengeodäsie, July 16~18, 2008, Bad Kötzing, Germany.
- Förstner, W. (1979), *Ein Verfahren zur Schätzung von Varianz- und Kovarianzkomponenten*, Allgemeine Vermessungsnachrichten, No. 11-12, 1979, pp. 446-453.
- Grafarend, E.; Kleusberg, A.; Schaffrin, B. (1980), *An Introduction to the Variance and Covariance Component Estimation of Helmert Type*, ZfV, Vol. 105, No. 4, 1980, pp. 161-180.
- Helmert, F. R. (1907), *Die Ausgleichungsrechnung nach der Methode der kleinsten Quadrate*, Zweite Auflage, Teubner, Leipzig.
- Hu, Congwei; Chen, Wu; Chen, Yongqi and Liu, Dajie (2003) *Adaptive Kalman Filtering for Vehicle Navigation*, Vol. 2, No. 1, 2003, Journal of Global Positioning Systems, pp. 42-47.
- Hu, Congwei and Liu, Dajie (2002), *Adaptive Kalman Filtering and Its Application Based on Variance Component Estimation*, Vol. 19, No. 1, 2002, Journal of Zhengzhou Institute of Surveying and Mapping, China (in Chinese).
- Koch, K.R. (1986), *Maximum Likelihood Estimate of Variance Components*, Bulletin Geodesique, Vol. 60, 1986, pp. 329-338.
- Li, Deren (1984), *Zur Lokalisierung grober Fehler mit Hilfe der Iterationsmethode mit variable Beobachtungsgewichten*, Commission III, 15th ISPRS Congress, Rio de Janeiro, 1984, pp. 83-93.
- Mehra, R. K.; Peschon, J. (1971), *An Innovations Approach to Fault Detection and Diagnosis in Dynamic Systems*, Automatica, Vol. 16, 1971, pp. 637-640.
- Milbert, Dennis (2008), *An analysis of the NAD 83 (NSRS2007) National Readjustment*, National Geodetic Survey (http://www.ngs.noaa.gov/PUBS_LIB/NSRS2007/).
- Rietdorf, Andreas (2004), *Automatisierte Auswertung und Kalibrierung von scanneden Messsystemen mit tachymetrischen Messprinzip*, PhD thesis, Civil Engineering and Applied Geosciences, Technical University Berlin, 2005.
- Sage, A.P.; Huas, G.W. (1969), *Adaptive Filtering with unknown Prior Statistics*, Proceedings of the 1969 JACC, 1969, pp. 760-769.
- Sieg, Detlef; Hirsch, Milo (2000), *Varianzkomponenten-schätzung in ingenieurgeodätischen Netzen*, AVN, No. 3, 2000, pp. 82-90.
- Tesmer, Volker (2004), *Das stochastische Modell bei der VLBI-auswertung*, PhD dissertation, No. 573, Reihe C, Deutsche Geodätische Kommission (DGK), Munich, 2004.
- Teunissen, P.J.G., Amiri-Simkooei, A.R.(2008), *Least-squares variance component estimation*, Journal of Geodesy, Vol. 82, No.2, 2008, pp. 65-82.
- Tiberius, Christian; Kenselaar, Frank (2003), *Variance component estimation and Precise GPS Positioning: Case Study*, Journal of Surveying Engineering, No. 11, February 2003.
- Tsang, W.; Glover, J. and Bach, R. (1981) *Identifiability of unknown noise covariance matrices for some special cases of a linear, time-invariant, discrete-time dynamic system*, Automatic Control, IEEE Transactions, Vol. 26, Issue: 4, pp. 970- 974.
- Wang, Jianguo (2008), *Test Statistics in Kalman Filter*, Journal of Global Positioning Systems, Vol. 7, No. 1, 2008, pp. 81-90.
- Wang, Jianguo (1997), *Filtermethoden zur fehlertoleranten kinematischen Positions-bestimmung*, Schriftenreihe Studiengang Vermessungswesen, Federal Arm-Forced University Munich, Germany, No. 52, Neubiberg, 1997.
- Wang, Jinling and Rizos, Chris (2002), *Stochastic Assessment of GPS Carrier Phase Measurements for Precise Static Relative Positioning*, Journal of Geodesy, Vol. 76, No. 2, 2002.

- Xu, Peiliang; Liu, Yumei; Shen, Yunzhong (2007), *Estimability analysis of variance and covariance components*, Journal of Geodesy, Vol. 81, No.9, 2007, pp. 593-602.
- Xu, Peiliang; Shen, Yunzhong; Fukuda, Yoichi; Liu, Yumei (2006), *Variance component estimation in linear inverse ill-posed models*, Journal of Geodesy, Vol. 80, No.2, 2006, pp. 69-81.
- Yu, Z.C.(1996), *A universal formula of maximum likelihood estimation of variance-covariance components*, Journal of Geodesy, Vol. 70, No. 4, 1996, pp. 233-240.
- Zhou, X.W.; Dai, W.J.; Zhu, J.J.; Li, Z.W. and Zou, Z.R. (2006), *Helmert Variance Component Estimation-based Vondrak Filter and its Application in GPS Multipath Error Mitigation*, VI Hotine-Marussi Symposium on Theoretical and Computational Geodesy, International Association of Geodesy Symposia, Volume 132, Springer Berlin Heidelberg, 2006.

The corresponding author

Dr.-Ing. Jian-Guo Wang (jgwang@yorku.ca) has been a faculty member in Geomatics Engineering at York University in Toronto, Canada since 2006. He worked as Senior Navigation Analyst at Applanix Corp. from 1999 to 2006. Prior to this, he obtained his Bachelor's Degree and Master's Degree in Surveying Engineering from Wuhan Technical University of Surveying and Mapping, and his Dr.-Ing. in Geomatics from University of The Federal Armed Forces Munich, Germany. With more than 20 years of academic and industrial experience in Geomatics in China, Germany and Canada, his current research interests focus on multisensor integrated positioning and navigation technology, high precision and high accuracy Engineering Surveying, and advanced data processing algorithms in Geomatics.